A set of multi-dimensional orthogonal basis functions and its application to solve integral equations

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Abstract: A new set of multi-dimensional orthogonal basis functions and some of their properties are introduced. These functions are extension of triangular functions (TFs) in $n$ dimensions. Expansion of multi-variable functions with respect to these functions is presented. Also, the relation of these new functions to the block-pulse functions (BPFs) in $n$ dimensions will be investigated.

Many applied problems are often discussed in $n$ dimensions, consequently the multi-dimensional moment method using the current orthogonal basis functions will be used to solve two-variable integral equations. The obtained results are compared with those of the multi-dimensional moment method using BPFs. These comparisons show efficiency and accuracy of the new orthogonal basis functions applied to solve multi-dimensional integral equations. Finally, a study of the representational error will be made to estimate the mean integral squared error for the TF approximation of a function $f(s, t)$ of Lebesgue measure.

Keywords: Multi-dimensional functions; Orthogonal basis functions; Triangular functions; Moment method; Multi-variable integral equations.

1 Introduction

Many physical and engineering problems can be formulated to the mathematical models. These models are often of the forms integral and integro-differential equations, differential
equations, dynamic systems, control systems, etc. There are many approaches to solve these problems.

In recent years, numerical methods using orthogonal basis functions have been recommended by many authors [7, 8, 10, 19, 20]. Some authors have used Sine-Cosine functions and orthogonal polynomials such as Chebyshev, Legendre, and Lagrange polynomials [2, 11, 12]. Piecewise constant or linear orthogonal functions such as block-pulse functions, Walsh functions, Haar functions, triangular functions [3–6, 9, 13–17, 23], also various wavelets [1, 11, 12, 13, 18] have been considered by many researchers for modeling the physical or engineering problems and to solve different functional equations, often producing very good accuracy.

Whereas many problems are occurred in \( n \) dimensions, therefore introducing new sets of multi-dimensional orthogonal basis functions and consequently illustrating numerical methods based on them can be necessary.

In this article, first of all, some characteristics of TFs are briefly described. Then, the new set of multi-dimensional orthogonal basis functions as a generalization of triangular functions is introduced. The characteristics of these new functions and the expansion of multi-variable functions with respect to them are presented. Also, the relation of presented functions to multi-dimensional BPFs is shown.

In section 4, multi-dimensional moment method using the new basis functions will be proposed. This method is applied to solve some two-variable integral equations. The obtained results are compared with those of the moment method using two-dimensional BPFs.

Finally, an estimation of the mean integral squared error will be presented.

2 Review of Triangular Functions

A special representation of one-dimensional triangular functions have been presented by Deb et al. [9] and studied and used by Babolian et al. [4, 5].

Two \( m \)-sets of triangular functions (TFs) are defined over the interval \([0, T]\) as [9]

\[
T_{1i}(t) = \begin{cases} 
1 - \frac{t - ih}{h}, & \text{if } ih \leq t < (i + 1)h, \\
0, & \text{otherwise,}
\end{cases} 
\]

\[
T_{2i}(t) = \begin{cases} 
\frac{t - ih}{h}, & \text{if } ih \leq t < (i + 1)h, \\
0, & \text{otherwise,}
\end{cases} 
\]

where \( i = 0, 1, \ldots, m - 1 \), with a positive integer value for \( m \). Also, consider \( h = T/m \), and \( T_{1i} \) as the \( i \)th left-handed triangular function and \( T_{2i} \) as the \( i \)th right-handed triangular function.

Assume that \( T = 1 \), so TFs are defined over \([0, 1]\), and \( h = 1/m \).

From the definition of TFs, it is clear that triangular functions are disjoint, orthogonal, and complete [9]. Therefore, we can write

\[
\int_0^1 T_{1i}(t)T_{1j}(t)dt = \int_0^1 T_{2i}(t)T_{2j}(t)dt = \begin{cases} 
\frac{h}{3}, & i = j, \\
0, & i \neq j.
\end{cases} 
\]
Also,
\[
\phi_i(t) = T_1_i(t) + T_2_i(t), \quad i = 0, 1, \ldots, m - 1,
\]
where \( \phi_i(t) \) is the \( i \)th block-pulse function defined as
\[
\phi_i(t) = \begin{cases} 
1, & ih \leq t < (i + 1)h, \\
0, & \text{otherwise,}
\end{cases}
\]
where \( i = 0, 1, \ldots, m - 1. \)

Consider the first \( m \) terms of left-handed triangular functions and the first \( m \) terms of right-handed triangular functions and write them concisely as \( m \)-vectors
\[
\mathbf{T}_1(t) = [T_1_0(t), T_1_1(t), \ldots, T_1_{m-1}(t)]^T,
\]
\[
\mathbf{T}_2(t) = [T_2_0(t), T_2_1(t), \ldots, T_2_{m-1}(t)]^T,
\]
where \( \mathbf{T}_1(t) \) and \( \mathbf{T}_2(t) \) are called left-handed triangular functions (LHTF) vector and right-handed triangular functions (RHTF) vector, respectively.

The expansion of a function \( f(t) \) over \([0, 1)\) with respect to TFs, may be compactly written as
\[
f(t) \approx \sum_{i=0}^{m-1} c_i T_1_i(t) + \sum_{i=0}^{m-1} d_i T_2_i(t)
\]
\[
= \mathbf{c}^T \mathbf{T}_1(t) + \mathbf{d}^T \mathbf{T}_2(t),
\]
where we may put \( c_i = f(ih) \) and \( d_i = f((i+1)h) \) for \( i = 0, 1, \ldots, m - 1. \) So, approximating a known function by TFs needs no integration to evaluate the coefficients.

In the next section, TFs will be generalized to \( n \) dimensions and a new set of multi-dimensional orthogonal basis functions will be introduced.

### 3 Multi-dimensional Triangular Functions

Here, TFs are generalized to \( n \) dimensions. We firstly define two-dimensional TFs, then these functions will be extended to \( n \) dimensions. Also, the expansion of multi-variable functions with respect to these new basis functions is presented and the relation of multi-dimensional BPFs to these functions will be shown.

#### 3.1 Definition and some characteristics

We define a set of two-dimensional triangular functions as \( T_{1_{i,j}}(s,t) \) and \( T_{2_{i,j}}(s,t) \) over a region \([0, T_1) \times [0, T_2)\), as follows:

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Figure 1: A set of two-dimensional LHTFs $T_{1i,j}(s,t)$ and RHTFs $T_{2i,j}(s,t)$, for $i, j = 0, 1$.

\[ T_{1i,j}(s,t) = \begin{cases} 1 - \frac{s + t - (ih_1 + jh_2)}{h_1 + h_2}, & \text{if } ih_1 \leq s < (i + 1)h_1, \text{ and } jh_2 \leq t < (j + 1)h_2, \\ 0, & \text{otherwise}, \end{cases} \tag{3.1} \]

\[ T_{2i,j}(s,t) = \begin{cases} \frac{s + t - (ih_1 + jh_2)}{h_1 + h_2}, & \text{if } ih_1 \leq s < (i + 1)h_1, \text{ and } jh_2 \leq t < (j + 1)h_2, \\ 0, & \text{otherwise}, \end{cases} \]

where $i = 0, 1, \ldots, m_1 - 1$ and $j = 0, 1, \ldots, m_2 - 1$, with positive integer values for $m_1, m_2$. Also, consider $h_1 = T_1/m_1$ and $h_2 = T_2/m_2$, and $T_{1i,j}$ as the $i, j$th left-handed triangular function (LHTF) and $T_{2i,j}$ as the $i, j$th right-handed triangular function (RHTF).

Figure 1 shows the set of two-dimensional orthogonal triangular basis functions, where for convenience, $m_1 = m_2$ has been arbitrarily chosen as 2 and $T_1 = T_2 = T$, so $h_1 = h_2 = h$.

Now, consider the first $m_1m_2$ terms of left-handed triangular functions and the first $m_1m_2$ terms of right-handed triangular functions and write them concisely as $m_1m_2$-
vectors

\[ \mathbf{T}_1(s, t) = [T_{1,0,0}(s, t), T_{1,0,1}(s, t), \ldots, T_{1,m_2-1}(s, t)], \ldots, T_{1,m_1-1,0}(s, t), \ldots, T_{1,m_1-1,m_2-1}(s, t)]^T, \]

\[ \mathbf{T}_2(s, t) = [T_{2,0,0}(s, t), T_{2,0,1}(s, t), \ldots, T_{2,m_2-1}(s, t), \ldots, T_{2,m_1-1,0}(s, t), \ldots, T_{2,m_1-1,m_2-1}(s, t)]^T, \]

(3.2)

where \( \mathbf{T}_1(s, t) \) and \( \mathbf{T}_2(s, t) \) are called left-handed triangular functions vector and right-handed triangular functions vector, respectively.

From definition (3.1), it is clear that these functions are disjoint, complete, and orthogonal; so that for \( s \in [0, T_1) \) and \( t \in [0, T_2) \), we can write

\[
\int_0^{T_1} \int_0^{T_2} T_{1,i,j}(s, t) T_{1,p,q}(s, t) ds dt
= \begin{cases}
\frac{h_1 h_2}{6(h_1 + h_2)^2} (2h_1^2 + 3h_1 h_2 + 2h_2^2), & i = p, j = q, \\
0, & \text{otherwise},
\end{cases}
\]

\[
\int_0^{T_1} \int_0^{T_2} T_{2,i,j}(s, t) T_{2,p,q}(s, t) ds dt
= \begin{cases}
\frac{h_1 h_2}{6(h_1 + h_2)^2} (2h_1^2 + 3h_1 h_2 + 2h_2^2), & i = p, j = q, \\
0, & \text{otherwise},
\end{cases}
\]

(3.3)

and

\[
\int_0^{T_1} \int_0^{T_2} T_{1,i,j}(s, t) T_{2,p,q}(s, t) ds dt
= \begin{cases}
\frac{h_1 h_2}{6(h_1 + h_2)^2} (h_1^2 + 3h_1 h_2 + h_2^2), & i = p, j = q, \\
0, & \text{otherwise}.
\end{cases}
\]

(3.4)

As a simple case, consider \( T_1 = T_2 = T \) and \( m_1 = m_2 \). Hence, \( h_1 = h_2 = h \) and the Eqs. (3.3) and (3.4) may be read as

\[
\int_0^{T} \int_0^{T} T_{1,i,j}(s, t) T_{1,p,q}(s, t) ds dt
= \begin{cases}
\frac{7h^2}{24}, & i = p, j = q, \\
0, & \text{otherwise},
\end{cases}
\]

(3.5)

\[
\int_0^{T} \int_0^{T} T_{2,i,j}(s, t) T_{2,p,q}(s, t) ds dt
= \begin{cases}
\frac{7h^2}{24}, & i = p, j = q, \\
0, & \text{otherwise},
\end{cases}
\]

and

\[
\int_0^{T} \int_0^{T} T_{1,i,j}(s, t) T_{2,p,q}(s, t) ds dt
= \begin{cases}
\frac{5h^2}{24}, & i = p, j = q, \\
0, & \text{otherwise}.
\end{cases}
\]

(3.6)

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Now, we extend the new set of orthogonal basis functions defined by Eqs. (3.1) to dimension $n$.

A system of $n$-dimensional TFs is defined as

\[
T_{1i_1,i_2,\ldots,i_n}(t_1,t_2,\ldots,t_n) = \begin{cases} 
1 - \frac{\sum_{k=1}^{n} t_k - \sum_{k=1}^{i_k} h_k}{\sum_{k=1}^{n} h_k}, & i_k h_k \leq t_k < (i_k + 1)h_k, \\
0, & \text{otherwise},
\end{cases} \tag{3.7}
\]

\[
T_{2i_1,i_2,\ldots,i_n}(t_1,t_2,\ldots,t_n) = \begin{cases} 
\sum_{k=1}^{n} t_k - \sum_{k=1}^{i_k} h_k, & i_k h_k \leq t_k < (i_k + 1)h_k, \\
0, & \text{otherwise},
\end{cases}
\]

where $t_k \in [0,T_k]$, $h_k = T_k/m_k$, and $i_k = 0,1,2,\ldots,m_k - 1$, for $k = 1,2,\ldots,n$. Also, $m_k$ denotes the number of segments on $[0,T_k)$ along the coordinate axis of the Cartesian direction $t_k$.

Note that in the case of $n$-dimensional, left-handed triangular functions vector and right-handed triangular functions vector can be defined as in Eqs. (3.2). But it should be noted that these vectors have $m_1 m_2 \ldots, m_n$ components. Ultimately, it can be easily shown that these functions are disjoint and orthogonal.

### 3.2 Expansion of multi-variable functions

Initially, the expansion of a two-dimensional function $f(s,t)$ with respect to the new set of TFs defined by Eqs. (3.1) is presented. This expansion may be compactly written as

\[
f(s,t) \simeq \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} c_{i,j} T_{1i,j}(s,t) + \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} d_{i,j} T_{2i,j}(s,t) \tag{3.8}
\]

where $c_{i,j}$ and $d_{i,j}$ are constant coefficients for $i = 0,1,\ldots,m_1 - 1$ and $j = 0,1,\ldots,m_2 - 1$, respectively and the $m_1 m_2$-vectors $c, d$ can be written as

\[
c^T = [c_0,0, c_0,1,\ldots, c_0,m_2-1, c_1,0,\ldots, c_1,m_2-1,\ldots, c_{m_1-1,0},\ldots, c_{m_1-1,m_2-1}]^T, \tag{3.9}
\]

\[
d^T = [d_0,0, d_0,1,\ldots, d_0,m_2-1, d_1,0,\ldots, d_1,m_2-1,\ldots, d_{m_1-1,0},\ldots, d_{m_1-1,m_2-1}]^T.
\]

The coefficients $c_{i,j}$ and $d_{i,j}$, for $i = 0,1,\ldots,m_1 - 1$ and $j = 0,1,\ldots,m_2 - 1$ can be set to

\[
c_{i,j} = f(ih_1, jh_2), \tag{3.10}
\]

\[
d_{i,j} = c_{i+1,j+1} = f((i+1)h_1, (j+1)h_2).
\]

So, approximating a known function by two-dimensional TFs may need no integration to evaluate the coefficients. It can be done only by sampling of two-variable function $f$ at $(m_1 + 1)(m_2 + 1)$ specific points on the region $[0,T_1] \times [0,T_2]$. 


The component \( d_{m_1-1,m_2-1} = f(m_1 h_1, m_2 h_2) = f(T_1, T_2) \) of vector \( \mathbf{d} \) can be computed if the function \( f(s, t) \) is defined over \([0, T_1] \times [0, T_2]\). If this function is defined over \([0, T_1] \times [0, T_2]\) and be a periodic function, we assume \( f(T_1, T_2) = f(0, 0) \).

Also, it should be noted that the above computations of coefficients are not an optimal case. If we consider an optimal representation of \( f(s, t) \) leading to least mean-absolute error, the coefficients \( c_{i,j} \) and \( d_{i,j} \), for \( i = 0, 1, \ldots, m_1 - 1 \) and \( j = 0, 1, \ldots, m_2 - 1 \) are to be determined from the following two equations:

\[
\begin{align*}
\int_{ih_1}^{(i+1)h_1} \int_{jh_2}^{(j+1)h_2} f(s, t) T_{1,i,j}(s, t) \, ds \, dt &= c_{i,j} \int_{ih_1}^{(i+1)h_1} \int_{jh_2}^{(j+1)h_2} [T_{1,i,j}(s, t)]^2 \, ds \, dt \\
&\quad + d_{i,j} \int_{ih_1}^{(i+1)h_1} \int_{jh_2}^{(j+1)h_2} T_{1,i,j}(s, t) T_{2,i,j}(s, t) \, ds \, dt, \\
(3.11)
\end{align*}
\]

\[
\begin{align*}
\int_{ih_1}^{(i+1)h_1} \int_{jh_2}^{(j+1)h_2} f(s, t) T_{2,i,j}(s, t) \, ds \, dt &= c_{i,j} \int_{ih_1}^{(i+1)h_1} \int_{jh_2}^{(j+1)h_2} T_{1,i,j}(s, t) T_{2,i,j}(s, t) \, ds \, dt \\
&\quad + d_{i,j} \int_{ih_1}^{(i+1)h_1} \int_{jh_2}^{(j+1)h_2} [T_{2,i,j}(s, t)]^2 \, ds \, dt.
\end{align*}
\]

Using orthogonality of two-dimensional TFs from Eqs. (3.3) and (3.4) we obtain

\[
\begin{align*}
A &= \frac{h_1 h_2}{6(h_1 + h_2)^2} (2h_1^2 + 3h_1 h_2 + 2h_2^2) c_{i,j} \\
&\quad + \frac{h_1 h_2}{6(h_1 + h_2)^2} (h_1^2 + 3h_1 h_2 + h_2^2) d_{i,j}, \\
B &= \frac{h_1 h_2}{6(h_1 + h_2)^2} (h_1^2 + 3h_1 h_2 + h_2^2) c_{i,j} \\
&\quad + \frac{h_1 h_2}{6(h_1 + h_2)^2} (2h_1^2 + 3h_1 h_2 + 2h_2^2) d_{i,j}, \\
(3.12)
\end{align*}
\]

where,

\[
\begin{align*}
A &= \int_{ih_1}^{(i+1)h_1} \int_{jh_2}^{(j+1)h_2} f(s, t) T_{1,i,j}(s, t) \, ds \, dt, \\
B &= \int_{ih_1}^{(i+1)h_1} \int_{jh_2}^{(j+1)h_2} f(s, t) T_{2,i,j}(s, t) \, ds \, dt, \\
(3.13)
\end{align*}
\]

where \( s \in [ih_1, (i + 1)h_1) \) and \( t \in [jh_2, (j + 1)h_2) \).

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To compare these two kinds of representations, an example will be illustrated in subsection 3.3.

Now, let \( f(t_1, t_2, \ldots, t_n) \) be an \( n \)-variable function. The expansion of \( f \) with respect to \( n \)-dimensional TFs has the following form:

\[
f(t_1, t_2, \ldots, t_n) \simeq \sum_{i_1=0}^{m_1-1} \sum_{i_2=0}^{m_2-1} \cdots \sum_{i_n=0}^{m_n-1} c_{i_1, i_2, \ldots, i_n} T_1(t_1, t_2, \ldots, t_n) \\
+ \sum_{i_1=0}^{m_1-1} \sum_{i_2=0}^{m_2-1} \cdots \sum_{i_n=0}^{m_n-1} d_{i_1, i_2, \ldots, i_n} T_2(t_1, t_2, \ldots, t_n),
\]

where \( c_{i_1, i_2, \ldots, i_n} \) and \( d_{i_1, i_2, \ldots, i_n} \) are constant coefficients for \( i_k = 0, 1, \ldots, m_k - 1 \), \( k = 0, 1, \ldots, n \), respectively. These coefficients can be set as follows:

\[
c_{i_1, i_2, \ldots, i_n} = f(i_1 h_1, i_2 h_2, \ldots, i_n h_n), \\
d_{i_1, i_2, \ldots, i_n} = c_{i_1+1, i_2+1, \ldots, i_n+1} = f((i_1 + 1) h_1, (i_2 + 1) h_2, \ldots, (i_n + 1) h_n).
\]

Corresponding to the two-dimensional case, it is clear that \( \mathbf{c} \) and \( \mathbf{d} \) are \( m_1 m_2 \ldots m_n \)-vectors and \( f(T_1, T_2, \ldots, T_n) \) can be set to a suitable value like the two-dimensional case. Also, it is clear that the optimal coefficients \( c_{i_1, i_2, \ldots, i_n} \) and \( d_{i_1, i_2, \ldots, i_n} \), can be easily computed using orthogonality of \( n \)-dimensional TFs like the two-dimensional case.

### 3.3 Multi-dimensional BPFs and comparing them with multi-dimensional TFs

In this section, the \( n \)-dimensional BPFs are briefly described, and their relation to multi-dimensional TFs is shown.

A system of \( n \)-dimensional BPFs over a region \( t_k \in [0, T_k] \), for \( k = 1, 2, \ldots, n \), is defined as \( \phi_{i_1, i_2, \ldots, i_n}(t_1, t_2, \ldots, t_n) \) to form an orthogonal basis for approximating a multi-dimensional square-integrable function \( f(t_1, t_2, \ldots, t_n) \). For each set of integers \( i_1, i_2, \ldots, i_n \), the \( n \)-dimensional BPF is defined as follows [21]:

\[
\phi_{i_1, i_2, \ldots, i_n}(t_1, t_2, \ldots, t_n) = \begin{cases} 1, & i_k h_k \leq t < (i_k + 1) h_k, \\ 0, & \text{otherwise}, \end{cases}
\]

where \( m_k \) denotes the number of segments on \( [0, T_k] \) along the coordinate axis of the Cartesian direction \( t_k \) and \( i_k = 0, 1, \ldots, m_k - 1 \), and \( h_k = T_k / m_k \), for \( k = 1, 2, \ldots, n \). It is clear that these functions are complete, disjoint and orthogonal.

The set of BPFs may be written as a vector \( \Phi(t_1, t_2, \ldots, t_n) \) of dimension \( \prod_{k=1}^{n} m_k \).

For instance, when \( n = 2 \):

\[
\Phi(t_1, t_2) = [\phi_{0,0}(t_1, t_2), \phi_{0,1}(t_1, t_2), \ldots, \phi_{0,m_2-1}(t_1, t_2), \ldots, \\
\phi_{m_1-1,0}(t_1, t_2), \ldots, \phi_{m_1-1,m_2-1}(t_1, t_2)]^T.
\]

A multi-dimensional square-integrable function in the region \( t_k \in [0, T_k] \), for \( k = 1, 2, \ldots, n \), may be approximated as
\[ f(t_1, t_2, \ldots, t_n) \simeq \sum_{i_1=0}^{m_1-1} \sum_{i_2=0}^{m_2-1} \cdots \sum_{i_n=0}^{m_n-1} f_{i_1,i_2,\ldots,i_n}(t_1, t_2, \ldots, t_n) \]

\[ = F^T \Phi(t_1, t_2, \ldots, t_n), \]

where \( f_{i_1,i_2,\ldots,i_n} \) are BPF coefficients, and \( F \) and \( \Phi \) are vectors with \( \prod_{k=1}^{n} m_k \) components. These coefficients may be obtained from the scalar products as

\[ f_{i_1,i_2,\ldots,i_n} = (f, \phi_{i_1,i_2,\ldots,i_n}(t_1, t_2, \ldots, t_n)) \]

\[ = \prod_{k=1}^{n} m_k \int_{i_1h}^{(i_1+1)h} \int_{i_2h}^{(i_2+1)h} \cdots \int_{i_nh}^{(i_n+1)h} f(t_1, t_2, \ldots, t_n) \, dt_1 dt_2 \cdots dt_n, \]

where \( t_k \in [i_kh_k, (i_k + 1)h_k) \).

As it has been shown in Eq. (3.19), computing BPF coefficients needs \( n \)-tiple integrations, but these computations for TFs can contain no integration.

Now, multi-dimensional basis TFs may be compared with BPFs. From their definitions, it can be easily concluded that both multi-dimensional TFs and BPFs are orthogonal, disjoint, and finite. As it was shown, BPFs are piecewise constant, but TFs are not, so approximating by BPFs provides staircase solution with more error than TFs in most cases. Note that analysis by TFs provides piecewise linear solution. Corresponding to Eqs. (3.15) and (3.19) we see that determination of any coefficient of multi-variable function \( f(t_1, t_2, \ldots, t_n) \) with respect to BPFs involves \( n \)-tiple integration of \( f \), but using TFs we can use only samples of \( f \) at specific points. Ultimately, both multi-dimensional BPFs and TFs can be easily extended over any region in space \( \mathbb{R}^n \), and like the BPFs, the multi-dimensional TFs can be easily normalized.

To compare the TFs with BPFs from the viewpoint of effectiveness, let us consider a two-variable function and approximate it with respect to two-dimensional BPFs and TFs. For simplicity, it is assumed \( T_k = 1 \) and \( m_k = m \), therefore \( s, t \in [0, 1) \) and \( h_k = h = 1/m \), for \( k = 1, 2 \). Hence, Eqs. (3.1) and Eq. (3.16) for two-dimensional TFs and BPFs may be read as

\[ T_{1,i,j}(s,t) = \begin{cases} 
1 - \frac{s+t-(i+j)h}{2h}, & \text{if } ih \leq s < (i+1)h, \\
0, & \text{if } jh \leq t < (j+1)h, \\
\text{otherwise}, & \end{cases} \]

\[ T_{2,i,j}(s,t) = \begin{cases} 
\frac{s+t-(i+j)h}{2h}, & \text{if } ih \leq s < (i+1)h, \\
0, & \text{if } jh \leq t < (j+1)h, \\
\text{otherwise}, & \end{cases} \]

and

\[ \phi_{i,j}(s,t) = \begin{cases} 
1, & \text{if } ih \leq s < (i+1)h, \\
0, & \text{if } jh \leq t < (j+1)h, \\
\text{otherwise}, & \end{cases} \]
where \( s, t \in [0, 1) \) and \( i, j = 0, 1, \ldots, m - 1 \).

Now, let \( f(s, t) = \exp(\frac{1}{2}(s + t)) \). For \( m = 2 \), the BPFs representation of \( f \) using Eqs. (3.18) and (3.19) is

\[
f_{BPF}(s, t) = [1.29073, 1.65733, 1.65733, 2.12805] \Phi(s, t).
\]

Similarly, optimal TFs representation of \( f \) from Eqs. (3.8), (3.12) and (3.13) is given by

\[
f_{TF-1}(s, t) = [0.96838, 1.24343, 1.24343, 1.59659] \mathbf{T}_1(s, t)
+ [1.61307, 2.07123, 2.07123, 2.65951] \mathbf{T}_2(s, t),
\]

and using Eqs. (3.8) and (3.10), non-optimal TFs representation of \( f \) is given by

\[
f_{TF-2}(s, t) = [1, 1.28403, 1.28403, 1.64872] \mathbf{T}_1(s, t)
+ [1.64872, 2.11700, 2.11700, 2.71828] \mathbf{T}_2(s, t).
\]

To better evaluation of the above representations, the mean-absolute error at points \((s, t)\), are computed as follows:

\[
E_n^{(m)} = \frac{1}{n} \sum_{i=1}^{n} |f(s_i, t_i) - f_m(s_i, t_i)|,
\]

where \( f(s, t) \) is the exact solution and \( f_m(s, t) \) is the approximate solution.

These mean-absolute errors at one hundred points of the forms \((s_i, t_j)\), for \( s_i = 0.1i \) and \( t_j = 0.1j \), for \( i, j = 0, 1, \ldots, 9 \) in \([0, 1) \times [0, 1)\), are \( 1.5E-1, 8.8E-3 \), and \( 4.2E-2 \), for BPFs, optimal TFs, and non-optimal TFs representations of \( f(s, t) \), respectively.

Also, Figs. 2(a), 2(b), 2(c), and 2(d) show the exact, BPFs, optimal TFs, and non-optimal TFs representations of \( f(s, t) \), respectively. Hence, it is apparent that TFs representation is superior to BPFs representation.

### 4 Multi-dimensional Moment Method

In this section, by using the results obtained in the previous section about multi-dimensional triangular functions as the orthogonal basis functions, an effective and very accurate method for solving multi-variable integral equations is presented.

For convenience, we consider two-variable Fredholm integral equations of the second kind. Note that the method presented here can be easily extended and applied to any multi-variable linear Fredholm and Volterra integral equations of the first and second kind.

As it was mentioned in previous section, two-dimensional TFs can be extended over any region such as \((s, t) \in [a, b) \times [c, d)\) in the plane \( \mathbb{R}^2 \). Therefore, consider the following Fredholm integral equation of the second kind with two variables:

\[
x(s, t) + \lambda \int_a^b \int_c^d k(s, t, s', t') x(s', t') ds' dt' = y(s, t),
\]

where \( k(s, t, s', t') \) is the kernel function, and \( \lambda \) is the constant parameter.
where \((s, t) \in [a, b) \times [c, d]\) and the functions \(k(s, t, s', t')\), \(y(s, t)\), and the parameter \(\lambda\) are known but \(x(s, t)\) is the unknown function to be determined.

Now, the moment method is presented for Eq. (4.1). Approximating the function \(x(s, t)\) with respect to the two-dimensional triangular functions by Eq. (3.8) gives

\[
x(s, t) \approx c^T T_1(s, t) + d^T T_2(s, t),
\]

(4.2)

where the \(m_1 m_2\)-vectors \(c\) and \(d\) are TFs coefficients of \(x(s, t)\) that should be determined.

Substituting Eq. (4.2) into (4.1) gives

\[
\sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} \left[ c_{i,j} \left( T_{1i,j}(s, t) + \lambda \int_a^b \int_c^d k(s, t, s', t') T_{1i,j}(s', t') ds' dt' \right) 
+ d_{i,j} \left( T_{2i,j}(s, t) + \lambda \int_a^b \int_c^d k(s, t, s', t') T_{2i,j}(s', t') ds' dt' \right) \right] \approx y(s, t).
\]

(4.3)

For computing the unknown \(m_1 m_2\)-vectors \(c\) and \(d\), we must consider \(2m_1 m_2\) appropriate points \((s_p, t_q)\), for \(s_p \in [a, b)\), and \(t_q \in [c, d)\). Substituting these points in Eq. (4.3) gives

\[
\sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2-1} \left[ c_{i,j} \left( T_{1i,j}(s_p, t_q) + \lambda \int_{s_{h_1}}^{(i+1) h_1} \int_{t_{h_2}}^{(j+1) h_2} k(s_p, t_q, s', t') T_{1i,j}(s', t') ds' dt' \right) 
+ d_{i,j} \left( T_{2i,j}(s_p, t_q) + \lambda \int_{s_{h_1}}^{(i+1) h_1} \int_{t_{h_2}}^{(j+1) h_2} k(s_p, t_q, s', t') T_{2i,j}(s', t') ds' dt' \right) \right] \approx y(s_p, t_q).
\]

(4.4)
Replacing $\approx$ with $=$ yields a linear system of $2m_1m_2$ algebraic equations for $2m_1m_2$ unknown components $c_{0,0}, \ldots, c_{0,m_2-1}, \ldots, c_{m_1-1,m_2-1}$ and $d_{0,0}, \ldots, d_{0,m_2-1}, \ldots, d_{m_1-1,m_2-1}$. Therefore, an approximate solution $x(s, t) \approx c^T T_1(s, t) + d^T T_2(s, t)$ is obtained for Eq. (4.1).

It is clear that one can apply this method for two-variable Fredholm integral equations of the first kind. Also, this method can be easily used to solve linear Volterra integral equations.

Finally, note that the above moment method uses optimal case of TF approximation. Therefore, there are $(m_1 + 1)(m_2 + 1)$ unknowns instead of $2m_1m_2$ unknowns because $d_{i,j} = c_{i+1,j+1}$. So, we need $(m_1 + 1)(m_2 + 1)$ collocation points over $[a, b] \times [c, d]$.

5 Numerical Examples

The moment method presented in this article, is applied to some examples. This approach uses both the new set of two-dimensional orthogonal basis TFs and two-dimensional BPFs, so the results obtained via two sets of basis functions can be compared. Additionally, these examples have been presented in [11] too, hence the numerical results obtained here can be compared with the results of another method.

The computations associated with the examples were performed using Matlab 7 on a Personal Computer.

Example 5.1. Consider the following two-variable Fredholm integral equation of the second kind [11]:

$$x(s, t) - \int_0^1 \int_0^1 (ss'^2 + t^2t')x(s', t') \, ds' \, dt' = y(s, t), \quad (5.1)$$

with the exact solution $x(s, t) = \exp(st)$, and suitable right hand side. For $m_1, m_2 = 10$, Fig. 3 shows the exact and approximate solutions. Also, the numerical results at ten specific points in $[0, 1] \times [0, 1]$ are presented in Table 1.

The mean-absolute errors from Eq. (3.25), at one hundred points of the forms $(s_i, t_j)$, for $s_i = 0.1i$ and $t_j = 0.1j$, for $i, j = 0, 1, \ldots, 9$ over $[0, 1] \times [0, 1]$, are $6.5E - 2$ and $5.6E - 3$, for BPFs and TFs approximate solutions, respectively.

Example 5.2. For the following two-variable linear Fredholm integral equation of the second kind [11]:

$$x(s, t) - \int_{-1}^1 \int_{-1}^1 \cos(ss') \cos(tt')x(s', t') \, ds' \, dt' = y(s, t), \quad (5.2)$$

where $y(s, t)$ is chosen so that Eq. (5.2) has the exact solution $x(s, t) = \sin(s + t)$, for $m_1, m_2 = 8$, Fig. 4 shows the exact and approximate solutions. Table 2 shows the numerical results at some specific points in $[-1, 1] \times [-1, 1]$ too.

The mean-absolute errors from Eq. (3.25), at one hundred points of the forms $(s_i, t_j)$, for $s_i = -1 + 0.2i$ and $t_j = -1 + 0.2j$, for $i, j = 0, 1, \ldots, 9$ over $[-1, 1] \times [-1, 1]$, are $6.3E - 2$ and $3.6E - 3$, for BPFs and TFs approximate solutions, respectively.
Table 1: Numerical results for Example 5.1

<table>
<thead>
<tr>
<th>(s,t)</th>
<th>Exact solution</th>
<th>Approximate solution using TFs</th>
<th>Approximate solution using BPFs</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.0)</td>
<td>1</td>
<td>0.997492</td>
<td>1.002428</td>
</tr>
<tr>
<td>(0.1,0.8)</td>
<td>1.083287</td>
<td>1.078875</td>
<td>1.134592</td>
</tr>
<tr>
<td>(0.2,0.4)</td>
<td>1.083287</td>
<td>1.079786</td>
<td>1.118388</td>
</tr>
<tr>
<td>(0.3,0.6)</td>
<td>1.197217</td>
<td>1.192452</td>
<td>1.254273</td>
</tr>
<tr>
<td>(0.4,0.3)</td>
<td>1.127497</td>
<td>1.123622</td>
<td>1.169745</td>
</tr>
<tr>
<td>(0.5,0.7)</td>
<td>1.419068</td>
<td>1.412062</td>
<td>1.508894</td>
</tr>
<tr>
<td>(0.6,0.9)</td>
<td>1.716007</td>
<td>1.705436</td>
<td>1.851894</td>
</tr>
<tr>
<td>(0.7,0.2)</td>
<td>1.150274</td>
<td>1.145733</td>
<td>1.265069</td>
</tr>
<tr>
<td>(0.8,0.1)</td>
<td>1.083287</td>
<td>1.078990</td>
<td>1.134748</td>
</tr>
<tr>
<td>(0.9,0.5)</td>
<td>1.568312</td>
<td>1.559442</td>
<td>1.684402</td>
</tr>
</tbody>
</table>

Figure 3: (a) The TF approximate solution. (b) The BPF approximate solution. (c) The exact solution.
Table 2: Numerical results for Example 5.2

<table>
<thead>
<tr>
<th>(s,t)</th>
<th>Exact solution</th>
<th>Approximate solution using TFs</th>
<th>Approximate solution using BPFs</th>
</tr>
</thead>
<tbody>
<tr>
<td>(−1,−1)</td>
<td>−0.909297</td>
<td>−0.939424</td>
<td>−0.983986</td>
</tr>
<tr>
<td>(−0.8,0.4)</td>
<td>−0.389418</td>
<td>−0.391667</td>
<td>−0.479426</td>
</tr>
<tr>
<td>(−0.6,−0.2)</td>
<td>−0.717356</td>
<td>−0.718223</td>
<td>−0.681639</td>
</tr>
<tr>
<td>(−0.4,0.6)</td>
<td>0.198669</td>
<td>0.198958</td>
<td>0.247404</td>
</tr>
<tr>
<td>(−0.2,0.8)</td>
<td>0.564642</td>
<td>0.571885</td>
<td>0.681639</td>
</tr>
<tr>
<td>(0,0)</td>
<td>0</td>
<td>0.005176</td>
<td>0.247404</td>
</tr>
<tr>
<td>(0.2,−0.8)</td>
<td>−0.564642</td>
<td>−0.571885</td>
<td>−0.681639</td>
</tr>
<tr>
<td>(0.4,0.2)</td>
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<td>0.567184</td>
<td>0.479426</td>
</tr>
<tr>
<td>(0.6,−0.6)</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(0.8,−0.4)</td>
<td>0.389418</td>
<td>0.391667</td>
<td>0.479426</td>
</tr>
</tbody>
</table>

Figure 4: (a) The TF approximate solution. (b) The BPF approximate solution. (c) The exact solution.
6 Error Analysis

The error analysis for BPF expansion of any square integrable function has been investigated by Rao [21]. Also, Deb et al. [9] have carried out an error analysis in the triangular function domain. Here, we present an error analysis for two-dimensional TF expansion of any square integrable function of Lebesgue measure.

Let us consider \( m_1 = m_2 = m \) and \( h_1 = h_2 = h \). The mean integral squared error in the \( i, j \)th region is given by

\[
[E_{ij}]^2 = \int_{ih}^{(i+1)h} \int_{jh}^{(j+1)h} \left[ \hat{f}(s,t) - f(s,t) \right]^2 \, ds \, dt, \quad (6.1)
\]

so that \( s \in [ih, (i+1)h) \), \( t \in [jh, (j+1)h) \), and \( \hat{f}(s,t) \) is the approximation of \( f(s,t) \) in the \( i, j \)th region and is defined as

\[
\hat{f}(s,t) = c_{i,j} T_{1_{i,j}} + d_{i,j} T_{2_{i,j}}, \quad (6.2)
\]

where \( c_{i,j} = f(ih,jh) \), \( d_{i,j} = f((i+1)h,(j+1)h) \). So,

\[
\hat{f}(s,t) = f(ih,jh) + m_{i,j} (s + t) - m_{i,j} (i + j) h, \quad (6.3)
\]

in which,

\[
m_{i,j} = \frac{f((i+1)h,(j+1)h) - f(ih,jh)}{2h}. \quad (6.4)
\]

Let the function \( f(s,t) \) be extended by first order Taylor series in the \( i, j \)th region around the point \( (\xi_i, \zeta_j) \). Hence,

\[
f(s,t) \approx f(\xi_i, \zeta_j) + (s - \xi_i) \hat{f}_s(\xi_i, \zeta_j) + (t - \zeta_j) \hat{f}_t(\xi_i, \zeta_j), \quad (6.5)
\]

where \( \hat{f}_s(\xi_i, \zeta_j) = \frac{\partial f(s,t)}{\partial s} \bigg|_{s=\xi_i, t=\zeta_j} \) and \( \hat{f}_t(\xi_i, \zeta_j) = \frac{\partial f(s,t)}{\partial t} \bigg|_{s=\xi_i, t=\zeta_j} \). Also, \( \xi_i \in [ih, (i+1)h] \), \( \zeta_j \in [jh, (j+1)h] \). Substituting Eqs. (6.3) and (6.5) into Eq. (6.1) gives

\[
[E_{ij}]^2 = \int_{ih}^{(i+1)h} \int_{jh}^{(j+1)h} \left[ f(ih,jh) + m_{i,j} (s + t) - m_{i,j} (i + j) h \right.
\]

\[
- f(\xi_i, \zeta_j) - (s - \xi_i) \hat{f}_s(\xi_i, \zeta_j) - (t - \zeta_j) \hat{f}_t(\xi_i, \zeta_j)] \, ds \, dt. \quad (6.6)
\]

Now consider

\[
A = f(ih,jh) - f(\xi_i, \zeta_j) + \xi_i \hat{f}_s(\xi_i, \zeta_j) + \zeta_j \hat{f}_t(\xi_i, \zeta_j) - m_{i,j} (i + j) h, \quad (6.7)
\]

\[
B = m_{i,j} - \hat{f}_s(\xi_i, \zeta_j),
\]

\[
C = m_{i,j} - \hat{f}_t(\xi_i, \zeta_j).
\]

Therefore, Eq. (6.6) can be written as

\[
[E_{ij}]^2 = \int_{ih}^{(i+1)h} \int_{jh}^{(j+1)h} [A + Bs + Ct]^2 \, ds \, dt. \quad (6.8)
\]
The Eq. (6.8) may be simplified to

$$[E_{i,j}]^2 = A^2h^2 + \frac{3i^2 + 3i + 1}{3}B^2h^4 + (2i + 1)ABh^3 + (2j + 1)ACH^3$$
$$+ \frac{(2i + 1)(2j + 1)}{2}BCh^4 + \frac{3j^2 + 3j + 1}{3}C^2h^4.$$  (6.9)

Then, the mean integral squared error over $m^2$ rectangles is given by

$$E^2 = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} [E_{i,j}]^2.$$  (6.10)

Let us now consider some special cases that can be derived from Eq.(6.9).

**Case I:** Assume that the function $f(s, t)$ is a constant function. Then, $\dot{f}_s(s, t) = \dot{f}_t(s, t) = 0$ and $m_{i,j} = 0$. Hence, $A = B = C = 0$ and Eq. (6.9) follows $[E_{i,j}]^2 = 0$. So, the mean integral squared error for constant functions is zero.

**Case II:** Let $f(s, t) = as + bt + c$, where $a$, $b$, $c$ are constant. Since $\dot{f}_s(s, t) = a$, $\dot{f}_t(s, t) = b$, and $m_{i,j} = \frac{a+b}{2}$, from Eqs. (6.7) we get $A = \frac{h}{2}(a-b)(i-j)$ and $B = -C = \frac{b-a}{2}$. So, from Eq. (6.9), the mean integral squared error is

$$[E_{i,j}]^2 = \frac{h^4}{24}(a-b)^2.$$  (6.11)

In this case, $[E_{i,j}]^2 = O(h^2)$, proportional to $h^2$. Also, it will be negligible if $|a-b|$ approaches to zero. It is clear that if $a = b$, then $[E_{i,j}]^2 = 0$. Therefore, the mean integral squared error for any function of the form $f(s, t) = a(s + t) + c$ is zero.

Here, the mean integral squared error has been obtained for two-dimensional functions. However, the presented procedure can be easily extended to obtain this error for arbitrary dimension.

### 7 Conclusion

This article introduced a new set of multi-dimensional orthogonal basis functions and its characteristics. Its effectiveness for representation of functions was established for optimal and non-optimal cases of multi-dimensional TFs. Also, their relation to the well-known BPFs was shown.

Then, the multi-dimensional moment method based on the new set of orthogonal basis functions was proposed. This approach, transforms a multi-variable integral equation to a set of algebraic equations. Its applicability and accuracy was checked on two examples. In these examples the approximate solution was briefly compared with exact and BPF approximate solutions only at ten specific points. These points have been selected over a region in $\mathbb{R}^2$ at random. But the mean-absolute errors have been computed for one hundred points in the related region. It follows from the figures, tables and mean-absolute errors that the accuracy of the obtained solutions using multi-dimensional TFs is quite good in comparison with BPF approximate solutions. Increasing the number of TFs decreases the
error of the solution rapidly. To show the convergence and stability of this approach, the current method can be run with increasing $m_1$ and $m_2$ until the computed results have appropriate accuracy.

[11] proposes a computational meshless method to solve Examples 5.1 and 5.2. Comparing figures obtained by current method using multi-dimensional TFs with figures in [11], it seems that the presented method is more accurate than the proposed approach in [11].

Finally, the mean integral squared error estimated in section 6 shows that for some cases the representational error is zero.

References


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