A new approach to Backlund Transformations of Burger Equation Arising in Longitudinal Dispersion of Miscible Fluid Flow through Porous Media

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Abstract:
In this paper a theoretical model is developed for the dispersion problem in porous media in which the flow is one dimensional and the average flow is unsteady. The Solution of the dispersion problem is presented by means of a new approach to \textit{Bäcklund} transformations of nonlinear partial differential equations.

Key Words: Miscible fluid, Burger equation, \textit{Bäcklund} Transformation, Dispersion, Concentration, Flows in porous media.

1 Introduction

The problem of miscible displacement can be observed in coastal areas, where the fresh water beds are gradually displaced by sea water. These day’s efforts are being made by the environmentalist to dispose the atomic waste products born from nuclear reactor and dumped inside the ground by using the same phenomenon of displacement. Among Many flow problems in porous media, one involves fluid mixtures called miscible fluids. A miscible fluid is a single phase fluid consisting of several completely dissolved homogenous fluid species, a distinct fluid-fluid interface doesn’t exist in a miscible fluid. The flow of miscible fluid is an important topic in petroleum industry; an enhanced recovery technique in oil reservoir involves injecting a fluid (solvent) that will dissolve the reservoir’s oil.

In a miscible displacement process a fluid is displaced in a porous medium by another fluid that is miscible with the first fluid. Miscible displacement in porous media plays a prominent role in many engineering and science fields such as oil recovery in petroleum engineering, contamination of ground water by waste product disposed under ground movement of mineral in the soil and recovery of spent liquors in pulping process.
These problems of dispersion have been receiving considerable attention from chemical, environmental and petroleum engineers, hydrologists, mathematicians and soil scientists. Most of the works reveal common assumption of homogenous porous media with constant porosity, steady seepage flow velocity and constant dispersion coefficient. For such assumption Ebach and White [5] studied the longitudinal dispersion problem for an input concentration that varies periodically with time and Ogata and Banks [13] for a constant input concentration. Hoopes and Herteman [6] studied the problem of dispersion in radial flow from a well fully penetrating, homogenous, isotropic non adsorbing confined aquifers. Bruce and Street [2] considered both longitudinal and lateral dispersion with in semi infinite non adsorbing porous media in a steady unidirectional flow fluid for a constant input concentration. Marino [9] considered the input concentration varying exponentially with time. Al-Niami and Rushton [1] and Marino [10] studied the analysis of flow against dispersion in porous media. Basak [3] presents an analytical solution to the problem of Evaporation from a horizontal soil column in which diffusivity increases linearly with moisture content and also to a problem of concentration dependent diffusion with decreasing concentration at the source. Hunt [7] applied the perturbation method to longitudinal and lateral dispersion in no uniform seepage flow through heterogeneous aquifers. Wang [16] discussed the concentration distribution of a pollutant arising from an instantaneous point source in a two dimensional water channel with non uniform velocity distribution. He employed Gill’s method to solve the convective diffusion equation. Kumar [8] discussed the Dispersion of Pollutants in Semi-Infinite Porous Media with Unsteady Velocity Distribution.

The present paper discusses the analytical solution of the nonlinear differential equation for longitudinal dispersion phenomena which takes places when miscible fluids mix in the direction of flow. The mathematical formulation of the problem yields a non linear partial differential equation. Solution has been obtained by using Bäcklund transformation.

The paper is organised in the following way:

The Bäcklund Transformation is introduced in section 1. In section 2 formulation of the model and the technique is applied on the dispersion problem to show the efficiency of the proposed approach. Section 3 ends this works with a brief conclusion.

2 Backlund transformation

Backlund transformation and Lax pairs play an important role in solitary theory. Because nonlinear iterative principle from Backlund transformations convert the problem of solving nonlinear differential equations to purely algebraic operation. Moreover, many connections among Bäcklund transformation, infinite conservation law and inverse scattering, etc. can be found by using Bäcklund transformations [4, 15]. Lax pairs convert nonlinear differential equations to a pair of linear equations.

Weiss, Tabor and Carnevale [17] defined the Painleve property for partial differential equations. Bäcklund Transformation, Lax pairs are obtained by truncating the expansion. To be precise, if the singularity manifold is determined by \( \phi(x, t) = 0 \) and \( u(x, t) = 0 \) is a solution of the partial differential equation

\[
u_t = K(u, u_x, \ldots)
\]

(2.1)

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Then suppose that
\[ u(x, t) = \frac{1}{\phi^\alpha} \sum_{j=0}^{\infty} u_j(x, t) \phi^j \] (2.2)

Where \( \alpha \) is a positive integer. \( \phi(x, t), u_j(x, t) \) analytic functions in a neighborhood of the manifold \( \phi = 0 \). Substituting (2.2) into partial equation (2.1) determines the possible \( \alpha \) and the recursion relations for \( u_j, j = 0, 1, 2, \ldots \). Backlund Transformations can be obtained by truncating expansion.

The main steps of our method are as follows. Suppose that the solution for differential equation (2.1) is of the form
\[ u = \frac{\partial^\alpha}{\partial x^\alpha} f(\phi) + u_1 \] (2.3)

Where \( u_1 \) a solution of (2.1) is also, \( f \) is determined later, \( \alpha \) is a positive integer.

1. Substituting (2.3) into (2.1) determines the possible \( \alpha \) through requiring the highest equality degree in nonlinear terms and a highest order partial derivative terms.

2. Substituting (2.3) into (2.1) Collecting all terms with the highest degree of \( \phi_x \) and setting its coefficient to zero, we obtain an ordinary equation then \( f(\phi) \) can be determined.

3. Collecting all terms with the same order derivatives off and setting their coefficients to zero respectively, then the compatibility conditions can be obtained.

3 Mathematical formulation and solution of the problem

The problem is to find the concentration as a function of time ‘t’ and position ‘x’ as the two miscible fluid flow through porous media on either sides of the mixed region. The single fluid equation describes the motion of fluid [14]. Here the mixing takes place longitudinally as well as transversely at \( t = 0 \) and a dot of fluid having \([C_0]\) concentration is injected over the phase. The dot moves in the direction of flow as well as perpendicular to the flow. Finally it takes the shape of the ellipse with a different concentration \([C_n]\) (fig-1).

According to Darcy’s law the equation of continuity for the mixture in case of incompressible fluids is given by
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \] (3.1)

Where ‘\( \rho \)’ is the density for mixture and ‘\( \vec{v} \)’ is the pore seepage velocity.

The equation of diffusion for a fluid flow through a homogeneous porous medium with out increasing or decreasing the dispersing material is given by
\[ \frac{\partial C}{\partial t} + \nabla \cdot (C \vec{v}) = \nabla \cdot \left[ \rho D \nabla \left( \frac{C}{\rho} \right) \right] \] (3.2)
Figure 1: Dispersion of an instantaneous point source in a uniform flow field

Where 'C' is the concentration of a fluid in a porous media. $D$ is the Coefficient of dispersion with nine components $D_{ij}$. In a laminar flow for an Incompressible fluid through homogeneous porous medium, density $\rho$ is constant. Then equation (3.2) becomes,

$$\frac{\partial C}{\partial t} + \bar{v}.\nabla C = \nabla . (D\nabla C)$$

(3.3)

Let us assume that the seepage velocity $\bar{v}$ is along the x- axis, then $\bar{v} = u(x, t)$ and the non zero components will be $D_{11} \approx D_L = \gamma$ (coefficient of longitudinal dispersion) and other Components will be zero [12].

Equation (3.3) becomes

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} = \gamma \frac{\partial^2 C}{\partial x^2}$$

(3.4)

Where $u$ is the component velocity along x-axis which is time dependent as well as concentration along x axis in $x \geq 0$ direction and $D_L > 0$ and it is the cross sectional flow velocity in porous media. $u = \frac{C(x,t)}{C_0}$, Where $x > 0$ and for $C_0 \approx 1$ by [11]. Equation (3.4) becomes

$$\frac{\partial C}{\partial t} + C \frac{\partial C}{\partial x} - \gamma \frac{\partial^2 C}{\partial x^2} = 0$$

(3.5)
Which is the non-linear Burger’s equation for longitudinal dispersion of miscible fluid flow through porous media.

The theory that follows is confined to dispersion in unidirectional seepage flow through semi-infinite homogeneous porous media. The seepage flow velocity is assumed unsteady. The dispersion systems to be considered are subject to an input concentration of contaminants $C_0$. The porous medium is considered as nonadsorbing. To obtain the solution to an unsteady flow problem a new approach to Bäcklund transformations of nonlinear evolution equation is presented.

Consider the input concentration is $C_0$. The governing partial differential equation (3.5) for longitudinal hydrodynamic dispersion with in a semi-infinite nonadsorbing porous medium in a unidirectional flow field in which $D$ is the longitudinal dispersion coefficient, $C$ is the average cross-sectional concentration, $u$ is the unsteady seepage velocity, $x$ is a coordinate parallel to flow and $t$ is time.

The initial and boundary conditions are

$$C(0,t) = C_0, t \geq 0; C(l,t) = C_1, t \geq 0 \quad \text{Provided } C_1 < C_0 \quad (3.6)$$

Suppose that its solution of (3.5) is of the form

$$C = \frac{\partial^\alpha}{\partial x^\alpha} f(\phi) + C_1(x,t) \quad (3.7)$$

Requiring the equality highest degree of $\phi_x$ in $CC_x$ and $-C_{xx}$, we find $\alpha = 1$, therefore

$$C = f(\phi) \phi_x + C_1(x,t) \quad (3.8)$$

Substituting (3.8) into (3.7) and calculating, we have

$$C_t + CC_x - \gamma C_{xx} = (f'f'' - \gamma f'''(\phi_x^3 + (f''\phi_x + f'\phi_x^2 + C_1f''\phi_x^2 - 3\gamma f''\phi_x\phi_{xx}) + (\phi_{tx} + C_1\phi_x + C_1\phi_{xx} - \gamma \phi_{xxx})f' + (C_1t + C_1C_1x - \gamma C_{1xx}) = 0 \quad (3.9)$$

Setting

$$f'f'' - \gamma f''' = 0 \quad (3.10)$$

Which has a solution

$$f = -2\gamma \log \phi \quad (3.11)$$

Thereby

$$f'^2 = 2\gamma f'' \quad (3.12)$$

Substituting (3.11) into (3.9) and using (3.10) we obtain

$$C_t + CC_x - C_{xx} = (\phi_t \phi_x + C_1\phi_x^2 - \gamma \phi_x \phi_{xx})f' + \frac{\partial}{\partial x}(\phi_t + C_1\phi_x - \gamma \phi_{xx})f' + C_1t + C_1C_1x - \gamma C_{1xx} = 0$$

Setting the coefficient of $f'$, $f''$ and final linear combination term of $C_1$ to zero gives

$$\phi_x(\phi_t + C_1\phi_x - \gamma \phi_{xx}) = 0 \quad (3.13)$$
\[
\frac{\partial}{\partial x}(\phi_t + C_1 \phi_x - \gamma \phi_{xx}) = 0 \tag{3.14}
\]

\[
C_{1t} + C_1 C_{1x} - \gamma C_{1xx} = 0 \tag{3.15}
\]

Due to \(\phi_x \neq 0\), we get from (3.13) \(\phi_t + C_1 \phi_x - \gamma \phi_{xx} = 0\)

Compatibility condition (3.14) is satisfied. Substituting (3.11) in to (3.8), we obtain Bäcklund transformation

\[
C = -2\gamma \frac{\phi_x}{\phi} + C_1 \tag{3.16}
\]

Where \(\phi\) satisfies (3.16), \(C_1\) satisfies (3.15).

If \(C_1 = 0\), we obtain Cole-Hopf transformation

\[
C = -2\gamma \frac{\phi_x}{\phi} \tag{3.17}
\]

If \(C_1 = \phi\), we obtain auto Bäcklund transformation

\[
C = -2\gamma \frac{\phi_x}{\phi} + \phi
\]

Equation (3.17) reduces (3.5) into diffusion equation.

\[
\frac{\partial \phi}{\partial t} = \gamma \frac{\partial^2 \phi}{\partial x^2} \tag{3.18}
\]

Let \(\phi = \phi (x, t)\) be a solution of Homogenous Parabolic Diffusion equation (3.18) with nonzero initial condition

\[
\phi(0, t) = \frac{2C^*_0}{C_0} \quad \phi(l, t) = \frac{2C^*_1}{C_1} \tag{3.19}
\]

By using Similarity Transformation

\[
\phi = g(\eta) \quad \eta = \frac{x}{\sqrt{4\gamma t}}
\]

Equation (3.18) becomes

\[
-\frac{dg}{d\eta} \eta = \frac{1}{2} \frac{d^2 g}{d\eta^2}
\]

The solution of (3.18) is

\[
g(\eta) = \frac{2C^*_0}{C_0} + k \int_0^\eta e^{-\eta^2} d\eta
\]

with initial condition

\[
g(0) = \frac{2C^*_0}{C_0} \tag{3.20}
\]
Using condition (3.20) we get
\[ k = \frac{2 \left( \frac{C^*}{C^*_1} - \frac{C^*_0}{C_0} \right)}{\int_0^l e^{-\eta^2} d\eta} \]

Therefore we get
\[ \phi = 2 \left( \frac{C^*_0}{C_0} + \frac{\left( \frac{C^*_1}{C^*_0} - \frac{C^*_0}{C_0} \right)}{\int_0^l e^{-\eta^2} d\eta} \right) \int_0^l e^{-\eta^2} d\eta \]

Where \( \eta = \frac{x}{\sqrt{4\gamma t}} \) (3.21)

Equation (3.21) and (3.5) gives
\[ C(x, T) = \frac{\partial}{\partial x} \left[ 2 \left( \frac{C^*_0}{C_0} + \frac{\left( \frac{C^*_1}{C^*_0} - \frac{C^*_0}{C_0} \right)}{\int_0^l e^{-\eta^2} d\eta} \right) \right] \int_0^l e^{-\eta^2} d\eta \]

Where \( \eta = \frac{x}{\sqrt{4\gamma t}} \) (3.22)

is the Concentration ‘C’ of porous medium.

## 4 Conclusion

Mathematical solutions have been developed for predicting the possible concentration of a given dissolved substance in unsteady unidirectional seepage flows through semi-infinite, homogeneous, isotropic porous media subject to the source concentrations that vary exponentially with time. Also the expressions taken into account the mass transfer from liquid matrix to solid matrix due to adsorption. The analytical expressions obtained here are useful to the study of salinity intrusion in groundwater, helpful in making quantitative predictions on the possible contamination of groundwater supplies resulting from groundwater movement through buried wastes.

## 5 NOMENCLATURE

\( C_0 \) - Initial Concentration of solute in liquid phase.
\( C \) - Concentration of solute in liquid phase.
\( \rho \) - density of the fluid
\( \bar{v} \) - pore seepage velocity
\( D \) - Dispersion coefficient based on \( u \).
\( t \) - time(s)
\( x \) - linear coordinate (m)

## References


