Application of Hes variational approach method for periodic solution of strongly nonlinear oscillation problems

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Abstract:
Applications of He’s Variational Approach Method (VAM) to solve the nonlinear oscillations is discussed in this paper. We established approximate analytical formulas for the period periodic solution. In contrast with conventional methods, in VAM, only one iteration leads to high accuracy towards solutions. The attained results are logical for the whole solution domain with high preciseness. We find that VAM is excellently for the whole range of initial amplitudes. VAM is in an excellent agreement of the periodical solutions with the Exact or other analytical solutions which has been demonstrated and discussed.

Keywords: He’s Variational Approach; Nonlinear Oscillation; Mixed Parity Nonlinear; Ball-Bearing Oscillating; Weakly Nonlinear System.

1 Introduction

Nonlinear analytical techniques for solving non-natural problems have been dominated by different methods. Investigate of nonlinear problems which are arisen in many areas of physics and also engineering is very important for scientists.
Various systematic procedures are utilized for solving nonlinear oscillation systems by many authors. Some of these well-known methods such as perturbation techniques [1–9] have their own limitations. To overcome the inadequacies, new techniques have been appeared in open literature, for example: delta-Perturbation Method [10–12], Artificial Parameter Method [13], Parameterized Perturbation Method [14], Perturbation Incremental Method [15], Parameter-Expanding (Expansion) Method [16–17], Lindstedt-Poincare Method [18–20], Variational Iteration Method [21–27], Harmonic Balance Method [28–31], Energy Balance Method [32–40] and also Variational Approach [41–46]. Among these methods, Variational Approach Method (VAM) has been considered to solve the nonlinear systems in this paper. By extending the Variational approach proposed by He, It has been established approximate analytical formulas for the period and periodic solution.

For this sake, the paper has been organized as follows: We describe basic idea of He’s variational approach method in Section 2; we will investigate applications of He’s variational approach method, to illustrate the applicability and accuracy of the method, we applied three examples. In section 4, some comparisons between analytical and numerical solutions are presented; eventually the last section contains one of the most significant findings of the paper.

2 Description of Hes Variational Approach Method

He[46] suggested a variational approach which is different from the known variational methods in open literature. Hereby a brief introduction of the method is given as follows:

\[ u'' + f(u) = 0 \]  \hspace{1cm} (2.1)

Its variational principle can be established using the semi-inverse method [46]:

\[ J(u) = \int_0^{T/4} \left( -\frac{1}{2} u'^2 + F(u) \right) dt \]  \hspace{1cm} (2.2)

Where T is period of the nonlinear oscillator, \( \frac{\partial F}{\partial u} = f \). Assume that its solution can be expressed as:

\[ u(t) = A \cos(\omega t) \]  \hspace{1cm} (2.3)

Where A and \( \omega \) are the amplitude and frequency of the oscillator, respectively. Substituting Eq.2.3 into Eq.2.2 results in:
\[ J(A, \omega) = \int_0^{T/4} \left( -\frac{1}{2} A^2 \omega^2 \sin^2 \omega t + F(A \cos \omega t) \right) dt \]
\[ = \frac{1}{\omega} \int_0^{\pi/2} \left( -\frac{1}{2} A^2 \omega^2 \sin^2 \omega t + F(A \cos \omega t) \right) dt \]
\[ = -\frac{1}{2} A^2 \omega \int_0^{\pi/2} \sin^2 t dt + \frac{1}{\omega} \int_0^{\pi/2} F(A \cos \omega t) dt \]  

(2.4)

Applying the Ritz method, we require:

\[ \frac{\partial J}{\partial A} = 0, \quad (2.5) \]

\[ \frac{\partial J}{\partial \omega} = 0 \quad (2.6) \]

But with a careful inspection, for most cases we find that

\[ \frac{\partial J}{\partial \omega} = -\frac{1}{2} A^2 \int_0^{\pi/2} \sin^2 t dt - \frac{1}{\omega^2} \int_0^{\pi/2} F(A \cos \omega t) dt < 0 \]  

(2.7)

Thus, we modify conditions 2.5 and 2.6 into a simpler form:

\[ \frac{\partial J}{\partial \omega} = 0 \]  

(2.8)

From which the relationship between the amplitude and frequency of the oscillator can be obtained.

3 APPLICATIONS

By applying He’s variational approach method, we will study the cyclic solutions to particular nonlinear oscillators. We consider Mixed parity nonlinear oscillator for example 1, then in example 2 we will solve equation of motion of a ball-bearing oscillating in a glass tube, and for example 3 we solve the weakly nonlinear system with discontinuities.

**Example 1**

**Mixed Parity Nonlinear Oscillator**

Hu in Ref. [31] presented improved solution using harmonic balance method. We consider mixed parity nonlinear oscillator to illustrate how to utilize variational approach method

\[ u'' + u + \varepsilon u^2 + u^3 = 0 \]  

(3.1)

With the boundary conditions of:
\[ u(0) = A, \ u'(0) = 0 \] (3.2)

Its variational formulation is:

\[ J(u) = \int_0^{T/4} \left( -\frac{1}{2}u'^2 + \frac{u^2}{2} + \varepsilon \frac{u^3}{3} + \frac{u^4}{4} \right) dt \] (3.3)

Choosing the trial function \( u(t) = A \cos(\omega t) \) into Eq. we obtain

\[ J(A) = \int_0^{T/4} \left( -\frac{1}{2}A^2 \omega^2 \sin^2 \omega t + \frac{1}{2}A^2 \cos^2 \omega t + \frac{1}{3} \varepsilon A^3 \cos^3 \omega t + \frac{1}{4} A^4 \cos^4 \omega t \right) dt \] (3.4)

The stationary condition with respect to \( A \) reads:

\[
\frac{\partial J}{\partial A} = \int_0^{\pi/2} \left( -A \omega^2 \sin^2 \omega t + A \cos^2 \omega t + A^2 \varepsilon \cos^3 \omega t + A^3 \cos^4 \omega t \right) dt = 0
\]

\[
= \int_0^{\pi/2} \left( -A \omega^2 \sin^2 t + A \cos^2 t + A^2 \varepsilon \cos^3 t + A^3 \cos^4 t \right) dt = 0
\]

\[ = -\omega^2 \int_0^{\pi/2} \sin^2 t dt + \int_0^{\pi/2} \cos^2 t dt + \varepsilon A \int_0^{\pi/2} \cos^3 t dt + A^2 \int_0^{\pi/2} \cos^4 t dt = 0 \] (3.5)

Solving Eq. , according to \( \omega \), with \( T = \frac{2\pi}{\omega} \):

\[ \omega = \sqrt{1 + \frac{8}{3\pi} \varepsilon A + \frac{3}{4} A^2} \] (3.6)

\[ T = \frac{2\pi}{\sqrt{1 + \frac{8}{3\pi} \varepsilon A + \frac{3}{4} A^2}} \] (3.7)

Comparing with results obtained by the harmonic balance method reported in Ref. [31], the above result is of high accuracy.

**Example 2**

**Motion of a Ball-Bearing Oscillating in a Glass Tube**

As an illustration, consider the motion of a ball-bearing oscillating in a glass tube that is bent into a curve such that the restoring force depends upon the cube of the displacement \( u \). The governing equation, ignoring frictional losses [47]:

\[ u'' + \varepsilon u^3 = 0 \] (3.8)

With the boundary conditions of:

\[ u(0) = A, \ u'(0) = 0 \] (3.9)
Its variational formulation is:

\[
J(u) = \int_{0}^{T/4} \left( -\frac{1}{2} u'^2 + \frac{\varepsilon u^4}{4} \right) dt
\]  
\tag{3.10}

Choosing the trial function \( u(t) = A \cos(\omega t) \) into Eq.3.10 we obtain

\[
J(A) = \int_{0}^{T/4} \left( -\frac{1}{2} A^2 \omega^2 \sin^2 \omega t + \frac{1}{4} \varepsilon A^4 \cos^4 \omega t \right) dt
\]  
\tag{3.11}

The stationary condition with respect to \( A \) reads:

\[
\frac{\partial J}{\partial A} = \int_{0}^{T/4} \left( -A \omega^2 \sin^2 \omega t + \varepsilon A^3 \cos^4 \omega t \right) dt = 0
\]
\[
= \int_{0}^{\pi/2} \left( -A \omega^2 \sin^2 t + \varepsilon A^3 \cos^4 t \right) dt = 0
\]
\[
= -\omega^2 \int_{0}^{\pi/2} \sin^2 t dt + \varepsilon A^2 \int_{0}^{\pi/2} \cos^4 t dt = 0
\]  
\tag{3.12}

Solving Eq.3.12, according to \( \omega \), with \( T = \frac{2\pi}{\omega} \):

\[
\omega = \sqrt{\frac{3}{4} \varepsilon A^2}
\]  
\tag{3.13}

\[
T = \frac{2\pi}{\sqrt{\frac{3}{4} \varepsilon A^2}} = \frac{7.255}{\varepsilon^{1/2} A}
\]  
\tag{3.14}

Its exact period can be readily obtained, which reads

\[
T = \frac{7.416}{\varepsilon^{1/2} A}
\]  
\tag{3.15}

It is obvious that the maximal relative error is less than 2.18%, and the obtained approximate period is valid for all \( \varepsilon > 0 \).

**Example 3**

**WEAKLY NONLINEAR SYSTEM**

We consider the weakly nonlinear system equation with discontinuities for this example; system having a single degree of freedom specifically [16]:

\[
u'' + u |u| = 0, \ u(0) = A, \ u'(0) = 0
\]  
\tag{3.16}

Where \( \varepsilon \) is a small dimensionless parameter, the dot denotes the derivative with respect to the dimensionless time \( t \) and \( u \) is a dimensionless dependent variable That, if \( u > 0 \) case(A) :

\[
u'' + u^2 = 0
\]  
\tag{3.17}

Therefore,
Table 1: Comparison of Variational Approach period with Exact period, for example 1 ($\varepsilon = 0.1$)

<table>
<thead>
<tr>
<th>A</th>
<th>Variational Approach period</th>
<th>Exact period</th>
<th>Error Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>229.4232</td>
<td>234.5145</td>
<td>2.170982</td>
</tr>
<tr>
<td>0.2</td>
<td>114.7116</td>
<td>117.2573</td>
<td>2.170982</td>
</tr>
<tr>
<td>0.4</td>
<td>57.35581</td>
<td>58.62863</td>
<td>2.170982</td>
</tr>
<tr>
<td>0.5</td>
<td>45.88465</td>
<td>46.9029</td>
<td>2.170982</td>
</tr>
<tr>
<td>5</td>
<td>4.588465</td>
<td>4.69029</td>
<td>2.170982</td>
</tr>
<tr>
<td>10</td>
<td>2.294232</td>
<td>2.345145</td>
<td>2.170982</td>
</tr>
<tr>
<td>100</td>
<td>0.229423</td>
<td>0.234515</td>
<td>2.170982</td>
</tr>
</tbody>
</table>

\[
J(u) = \int_{0}^{T/4} \left( -\frac{1}{2}u'^{2} + \frac{u^{3}}{3} \right) dt \tag{3.18}
\]

Substituting $u(t) = A \cos(\omega t)$ into Eq.3.18 we have:

\[
J(A) = \int_{0}^{T/4} \left( -\frac{1}{2}A^{2}\omega^{2}\sin^{2}\omega t + \frac{1}{3}A^{3}\cos^{3}\omega t \right) dt \tag{3.19}
\]

\[
\frac{\partial J}{\partial A} = \int_{0}^{T/4} (-A\omega^{2}\sin^{2}\omega t + A^{2}\cos^{3}\omega t) \ dt = 0
\]

\[
= \int_{0}^{\pi/2} (-A\omega^{2}\sin^{2}t + A^{2}\cos^{3}t) \ dt = 0
\]

\[
= -\omega^{2}\int_{0}^{\pi/2} \sin^{2}tdt + A\int_{0}^{\pi/2} \cos^{3}tdt = 0 \tag{3.20}
\]

Thus:

\[
\omega = \sqrt{\frac{8}{3\pi}}A = 0.921318\sqrt{A} \tag{3.21}
\]

To illustrate the remarkable accuracy of the obtained result, we compare the approximate frequency with the exact one, which reads

\[
\omega = 0.914681\sqrt{A} \tag{3.22}
\]

The accuracy arrives at 0.73%, which is remarkably good considering the simple guesses.

### 4 Results and Discussions

To illustrate and verify the accuracy of approximate analytical approach in example 1, a comparison between variational approach method and harmonic balance method
Figure 1: Comparison of the approximate solution VAM with Exact: (a) $A = 0.1$, (b) $A = 5$, (c) $A = 20$ (d) $A = 100$
is indicated. Comparing with results obtained by the harmonic balance method reported in Ref. [31], the above result is of high accuracy.

For example 2, a comparison of results with variational iteration method is presented in Eq.3.15, and also a comparison of percentage errors between variational approach and exact solution. The comparison of results between VAM and Exact solution for different parameter “A” is indicated in Table 1. It can be found that, the error percentages of variational approach method yield the perfect results with less than 2.18%, that approximate obtained solution by the proposed method is uniformly valid for any value of $\varepsilon$!

The comparison of different parameters via numerical and other approaches is presented in Figure 1 for example 3. The results are numerically obtained from Eq.2.3 for $u(t)$. For testing the accuracy, we considered He’s variational approach method (VAM) in 4 types: (a) $A=0.1$, (b) $A=5$, (c) $A=20$, (d) $A=100$. To illustrate the remarkable accuracy of the obtained result, we compare the approximate frequency with the exact solution. The accuracy arrives at 0.73%, which is remarkably good considering the simple guesses.

5 Conclusion

Variational Approach Method is utilized for 3 examples: mixed parity nonlinear oscillator, the motion of a ball-bearing oscillating in a glass tube and weakly nonlinear system with discontinuities. These examples have shown that the approximate analytical solutions are in excellent agreement with the corresponding exact solutions. The achieved results indicated that Variational approach is extremely simple, easy, powerful, and triggers good accuracy. Moreover, the method which is proved to be a powerful mathematical tool for studying of nonlinear oscillators.

References


