Exact travelling solutions for fifth order Caudrey-Dodd-Gibbon equation

M. Abdollahzadeh¹, M. Hosseini³, M. Ghanbarpour², H. Shirvani⁴

¹,²,⁴Babol University of Technology, Department of Mechanical Engineering, P. O. Box 484, Babol, Iran.
³Islamic Azad University, Ghaemshahr Branch, P. O. Box 163, Ghaemshahr, Iran

Abstract:
In this paper, we establish some distinct exact solutions for a nonlinear evolution equation. The sin-cosine method and the rational Exp-Function and sinh method are used to construct the travelling wave solutions of the fifth order Caudrey-Dodd-Gibbon (CDG) equation. These solutions may be important of significance for the explanation of some practical physical problem. The rational hyperbolic and other functions methods can be applied directly which the exact answers may have some physical interoperation. These characteristics make these methods so exceptional in exact solutions.

Keywords: Traveling wave solutions; sin-cosine method; Exponential rational function method; the rational sinh and cosh functions methods, the fifth order Caudrey-Dodd-Gibbon (CDG) equation

1 Introduction

Nonlinear evolution equations are widely used as models to describe complex physical phenomena and have a significant role in several scientific and engineering fields. These equations appear in solid state physics [1], fluid mechanics [2], chemical kinetics [3], plasma physics [4], population models, nonlinear optics, propagation of fluxons in Josephson junctions and etc... Analytical exact solutions to nonlinear partial differential equation play an important role in nonlinear science, since they can provide us much physical information and more inside into the physical aspects of the problem and thus lead to further applications. In recent years, quite a few methods for obtaining explicit travelling and solitary wave solutions of nonlinear evolutions equations have been proposed. A variety of powerful methods, such as inverse scattering method [5,6], bilinear transformation [7], Bcklund and Darboux transformation [7-11], the tanh-sech method [12,13,14], extended tanh method [15], Exp-function method[16-19], the sine-cosine method [20-22], the Jacobi elliptic function method [23-24], the \( \left( \frac{G'}{G} \right) \)-expansion method[25], F-expansion method [26,27], Li group analysis[28], He’s variational iteration method[29-31], He’s homotopy perturbation method[32-34] , homogeneous balance method [35,36] , adomian decomposition method [37-39] and so on. The sine–cosine method was developed by Wazwaz [22] and was successfully applied to nonlinear evolution equations [21, 40, 41, 42], to nonlinear equations systems [43].

Corresponding author: M. Hosseini — PROOF READING REQUIRED
In this paper we will apply the sine-cosine method and rational exponential function method to obtain the exact traveling wave solution of the following fifth order Caudrey-Dodd-Gibbon (CDG) equation

$$u_t + 30u_xu_{xx} + 30uu_{xxx} + 180u^2u_x + u_{xxxxx} = 0,$$  \hfill (1.1)

Its physical understanding was illustrated in [44], and its solitary solutions have been studied by many authors [45-50]. Wazwaz derived explicit travelling wave solutions using the tanh method in 2006[46] and multiple-soliton solutions using Hirota’s direct method combined with the simplified Hereman method in 2008 for the above equation.

2 Sine–cosine method

Wazwaz has summarized the main steps introduced for using sine–cosine method, as following:

1. Introduce the wave variables $\xi = x - ct$ into the PDE, we get

$$\phi(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, u_{xxx} \ldots) = 0$$  \hfill (2.1)

where $u(x, t)$ is travelling wave solution. This enables us to use the following changes:

$$u(x, t) = U(\xi),$$  \hfill (2.2)

$$\frac{\partial}{\partial t} = -c \frac{d}{d\xi}, \frac{\partial^2}{\partial t^2} = c^2 \frac{d^2}{d\xi^2}, \frac{\partial}{\partial x} = \frac{d}{d\xi}, \frac{\partial^2}{\partial x^2} = \frac{d^2}{d\xi^2}, \ldots$$  \hfill (2.3)

And so on for the other derivatives. Using (2.3) and (2.1), the nonlinear PDE ((2.1) ) changes to a nonlinear ODE:

$$\psi(U, -cU', c^2U'', U''' - cU', U'''' \ldots) = 0,$$  \hfill (2.4)

2. If all terms of the resulting ODE contain derivatives in $\xi$, then by integrating this equation, by considering the constant of integration to be zero, we obtain a simplified ODE.

3. By virtue of the technique of solution, we introduce the ansatz:

$$U(\xi) = u(x, t) = \begin{cases} 
\lambda \sin^\beta(\mu \xi), & |\mu \xi| < \pi \\
0, & otherwise
\end{cases}$$  \hfill (2.5)

Or

$$U(\xi) = u(x, t) = \begin{cases} 
\lambda \cos^\beta(\mu \xi), & |\mu \xi| < \frac{\pi}{2} \\
0, & otherwise
\end{cases}$$  \hfill (2.6)

where $\lambda$, $\mu$ and $\beta$ are parameters are to be determined later, $\mu$ and $c$ are the wave number and the wave speed, respectively, we use:

$$U(\xi) = \lambda \sin^\beta(\mu \xi),$$  \hfill (2.7a)

$$U^n(\xi) = \lambda^n \sin^{n\beta}(\mu \xi),$$  \hfill (2.7b)

$$(U^n)_\xi = n\mu \beta \lambda^n \cos(\mu \xi) \sin^{n\beta-1}(\mu \xi),$$  \hfill (2.7c)

$$(U^n)_\xi \xi = -n^2 \mu^2 \beta^2 \lambda^n \sin^{n\beta}(\mu \xi) + n\mu^2 \lambda^n \beta(n\beta - 1) \sin^{n\beta-2}(\mu \xi),$$  \hfill (2.7d)

$$U^n(\xi) = \lambda^n \mu^3 \beta(\mu \xi)^3 \sin^{n\beta-3}(\mu \xi) \cos^3(\mu \xi) + \lambda^n \mu^3 n\beta (3\beta n - 2) \sin^{n\beta-1}(\mu \xi) \cos(\mu \xi).$$  \hfill (2.7e)
and the derivatives of Eq. (2.6) becomes:

\[ U(\xi) = \lambda \cos^{\beta}(\mu \xi) \]  
\[ U^n(\xi) = \lambda^n \cos^{n\beta}(\mu \xi), \]  
\[ (U^n)_{\xi} = -n \mu \beta \lambda^n \sin(\mu \xi) \cos^{n\beta - 1}(\mu \xi), \]  
\[ (U^n)_{\xi \xi} = -n^2 \mu^2 \beta^2 \lambda^n \cos^{n\beta}(\mu \xi) + n \mu^2 \lambda^n \beta (n \beta - 1) \cos^{n\beta - 2}(\mu \xi), \]  
\[ (U^n)_{\xi \xi \xi} = n \lambda \mu^{3} \beta (\beta^2 + 2n \beta - 2) \sin^{n \beta - 3}(\mu \xi) \sin(\mu \xi) + \lambda^3 \mu^{2} n \beta (3 \beta n^2 - 2) \sin^{n \beta - 1}(\mu \xi) \sin(\mu \xi), \]

and so on for the other derivatives.

4. We substitute Eq. (2.7) or (2.8) into the reduced equation obtained above in (2.4), balance the terms of the cosine functions when (2.8) is used, or balance the terms of the sine functions when (2.7) is used, and solving the resulting system of algebraic equations by using the computerized symbolic calculations. We next collect all terms with same power in \( \cos^{k}(\mu \xi) \) or \( \sin^{k}(\mu \xi) \) and set to zero their coefficients to get a system of algebraic equations among unknowns \( \mu, \beta \) and \( \lambda \). We obtained all possible value of the parameters \( \mu, \beta \) and \( \lambda \).

3 The rational function in exp \( (\xi) \) method

The exp-function method was first proposed by He and Wu in 2006 [17] and systematically studied in [18, 19], and was successfully applied to KdV equation with variable coefficients [51], to high-dimensional nonlinear evolution equation [52], to burgers and combine KdV-mKdV (extended KdV) [53] equations, etc. In this section, we shall seek a rational function type of solution for a given partial equation, in terms of exp \( (\xi) \), of the following form:

\[ U = \sum_{k=0}^{m} \frac{a_k}{(1 + e^{\xi})^k} \]  

Where \( a_0, a_1, \ldots, a_m \) are some constants to be determined from the solution of (2.4).

Differentiating (3.1) with respect to \( \xi \), introducing the result into Eq. (2.4), and setting the coefficients of the same power of \( e^{\xi} \) equal to zero, we obtain algebraic equations. The rational function solution of the equation (2.1) can be solved by obtaining \( a_0, a_1, \ldots, a_m \) from this system.

4 The rational sinh and cosh functions methods

The rational sinh and cosh functions methods [54] can be expressed in the form

\[ U(\xi) = \frac{a_0 + b_0 \sinh^n(\mu \xi)}{1 + a_1 \sinh^m(\mu \xi)}, \quad \xi = x - ct, \quad n = 1, 2, \]  
\[ U(\xi) = \frac{a_0 + b_0 \cosh^n(\mu \xi)}{1 + a_1 \cosh^m(\mu \xi)}, \quad \xi = x - ct, \quad n = 1, 2, \]

where \( a_0, a_1, b_0, c \) and \( \mu \) are parameters that will be determined. The rational hyperbolic functions methods can be applied directly as assumed before. We then collect the coefficients of the resulting hyperbolic functions and setting it equal to zero, and solving the resulting equations to determine the parameters \( a_0, a_1, b_0, c \) and \( \mu \).
5 Exact travelling solutions for fifth order Caudrey-Dodd-Gibbon (CDG) equation

5.1 Application of Sine–cosine method

As described in Section 2, we make the transformation:

\[ u(x,t) = U(\xi), \quad \xi = x - ct, \quad (5.1) \]

Substituting Eq. (5.1) into (1.1) yields an ODE:

\[ -cU' + 30U'' + 30UU''' + 180U^2U'' + U^{(5)} = 0, \quad (5.2) \]

Integrating (5.2) and using the constants of integration to be zero, we find:

\[ -cU + 30UU'' + 60U^3 + U^{(4)} = 0, \quad (5.3) \]

Substituting Eq. (2.5) and Eq. (2.7) into (5.3) and rewriting the equation in terms of the sine function gives:

\[
\begin{align*}
(\lambda \beta^4 \mu^4 - c\lambda) \sin^3(\mu \xi) &- 30\lambda^2 \beta^4 \mu^2 \sin^2(\mu \xi) + 60\lambda^3 \sin^3(\mu \xi) \\
+ (4\lambda \beta^4 - 8\lambda \beta^2 \mu^4 + 6\lambda \beta^4 \mu^4 - 2\lambda \beta^4 \mu^4) \sin^2(\mu \xi) &+ (-6\lambda \beta^4 + \lambda \beta^4 \mu^4 + 11\lambda \beta^4 \mu^4) \sin(\mu \xi) \\
+ (-30\lambda^2 \beta^2 \mu^2 + 30\lambda^2 \beta^2 \mu^2) \sin^{2}(\mu \xi) &- (\mu \xi) = 0 \\
\end{align*}
\]

(5.4)

Balancing terms of the sine functions, we have:

\[
\begin{align*}
\beta(\beta - 1)(\beta - 2)(\beta - 3) &\neq 0 \quad (5.5a) \\
\beta - 4 = 3\beta &\Rightarrow \beta = -2 \quad (5.5b)
\end{align*}
\]

Substituting Eq. (5.5b) into (5.4) and equating the exponents and the coefficients of each pair of the cosine function, we obtain a system of algebraic equations:

\[
\begin{align*}
\sin^{-6}(\mu \xi) : 180\lambda^2 \mu^2 + 120\lambda \mu^4 + 60\lambda^3 & = 0 \quad (5.6a) \\
\sin^{-4}(\mu \xi) : -120\lambda \mu^2(\lambda + \mu^2) & = 0 \quad (5.6b) \\
\sin^{-4}(\mu \xi) \cdot \lambda(16\mu^4 - c) & = 0 \quad (5.6c)
\end{align*}
\]

Solving (5.6a)-(5.6c) with the aid of Maple, we obtain:

\[ \lambda = -2\mu^2, \quad c = -64\mu^6 \]

(5.7)

The results in Eq. (5.7) can be easily obtained if we also use the cosine method (2.6). Combining Eq. (5.7) with Eq. (2.5), the following periodic solutions will be obtained:

\[ u_1(x,t) = -\frac{\mu^2}{\sin^2(\mu(x - 16\mu^4 t))} = -\mu^2 \csc^2(\mu(x - 16\mu^4 t)), \quad 0 < \mu(x - 16\mu^4 t) < \pi \]

(5.8)

Eq. (5.8) will satisfy Eq. (1.1).

If we choose the following solution forms of Eq. (2.6) and insert them into Eq. (5.3), and rewriting the equation in terms of the cosine function gives:

\[
\begin{align*}
(-c\lambda + \lambda \beta^4 \mu^4) \cos(\mu \xi) &- 30\lambda^2 \beta^2 \mu^2 \cos^2(\mu \xi) + 60\lambda^3 \cos^3(\mu \xi) \\
+ (4\lambda \beta^4 + 6\lambda \beta^2 \mu^4 - 8\lambda \beta^4 \mu^4 - 2\lambda \beta^4 \mu^4) \cos^2(\mu \xi) &+ (-6\lambda \beta^4 \mu^4 + 11\lambda \beta^4 \mu^4 + \lambda \beta^4 \mu^4) \cos(\mu \xi) \\
+ (30\lambda^2 \beta^2 \mu^2 - 30\lambda^2 \beta^2 \mu^2 + 30\lambda^2 \beta^2 \mu^2) \cos^2(\mu \xi) &- (\mu \xi) = 0 \\
\end{align*}
\]

(5.9)
Balancing the terms of the cosine functions, we have:

\[ \beta(\beta - 1)(\beta - 2)(\beta - 3) \neq 0 \]  
\[ \beta - 4 = 3\beta \Rightarrow \beta = -2 \]  

(5.10a)  
(5.10b)

Substituting Eq. (5.10b) into Eq. (5.9) and equating the exponents and the coefficients of each pair of the cosine function, we obtain a system of algebraic equations:

\[ \sin^{-6}(\mu\xi) : 180\lambda^2\mu^2 + 120\lambda\mu^4 + 60\lambda^3 = 0, \]  
\[ \sin^{-4}(\mu\xi) : -120\lambda\mu^2(\lambda + \mu^2) = 0, \]  
\[ \sin^{-4}(\mu\xi) : \lambda(16\mu^4 - c) = 0 \]  

(5.11a)  
(5.11b)  
(5.11c)

Solving (5.11a)-(5.11c) with the aid of Maple, we obtain:

\[ \lambda = -2^2, c = 16\mu^4 \]  

(5.12)

Combining Eq. (5.12) with Eq. (2.6), the following periodic solution will be obtained:

\[ u_2(x, t) = -\frac{\mu^2}{\cos^2(\mu(x - 16\mu^4 t))} = -\mu^2 \sec^2(\mu(x - 16\mu^4 t)), |\mu(x - 16\mu^4 t)| < \frac{\pi}{2} \]  

(5.13)

### 5.2 Application of rational exponential function method

Now we shall seek a rational function type of solution to fifth order Caudrey-Dodd-Gibson (CDG) equation, in terms of \( \exp(\xi) \) in the form:

\[ u = U(\xi), \xi = \alpha(x - \beta t) \]  

(5.14)

Substituting Eq. (5.14) into Eq. (1.1) yields an ODE:

\[ -\epsilon U'' + 30U''U'' + 30U''U'' + 180U^2U' + U^{(2)} = 0, \]  

(5.15)

By use of the exp-function method, we may choose the solution of (5.15) in the form:

\[ U = a_0 + \frac{a_1}{1 + e^\xi} + \frac{a_2}{(1 + e^\xi)^2} \]  

(5.16)

Differentiating (5.16) with respect to \( \xi \), introducing the result into Eq. (5.15), and setting the coefficients of the same power of \( e^\xi \) equal to zero, we obtain these algebraic equations:

\[ 60\alpha a_0^3 + 180\alpha a_2^2 a_0 - \alpha^2 \beta a_2 + 180\alpha a_1^2 a_0 + 180\alpha a_1 a_2^2 - \alpha^2 \beta a_0 + 60\alpha a_1^3 - \alpha^2 \beta a_1 + 180\alpha a_2 a_0^2 + 60\alpha a_0^3 + 360\alpha a_1 a_2 a_0 + 180\alpha a_1^2 a_2 = 0 \]  

(5.17a)

\[ 180\alpha a_1 a_2^2 + 360\alpha a_0^3 - 90\alpha^3 a_1 a_2 - 4\alpha \beta a_2 - 60\alpha^3 \beta a_2 - 60\alpha^3 a_0 a_2 - \alpha^5 a_1 + 180\alpha a_1^3 \]  

\[ -5\alpha \beta a_1 - 30\alpha^3 a_1^2 + 1080\alpha a_1 a_2 a_0 + 720\alpha a_0^3 a_2 + 720\alpha a_1^2 a_0 + 360\alpha a_2^2 a_0 - 6\alpha \beta a_0 + 360\alpha a_2^2 a_0 - 30\alpha^3 a_0 a_1 + 900\alpha a_1^2 a_0 = 2\alpha^5 a_2 = 0 \]  

(5.17b)

\[ 180\alpha a_1^3 + 1800\alpha a_1 a_2^2 + 360\alpha a_0^3 + 6\alpha \beta a_2 + 900\alpha a_0^3 + 180\alpha a_0^2 a_2 + 1080\alpha a_2^2 a_0 \]  

\[ +120\alpha^3 a_2^2 + 60\alpha^3 a_1 a_2 + 1080\alpha a_2 a_0^2 - 15\alpha \beta a_0 + 10\alpha^5 a_1 - 30\alpha^3 a_1^2 + 1080\alpha a_1 a_2 a_0 \]  

\[ -10\alpha \beta a_1 + 180\alpha a_2^2 a_0 - 60\alpha^3 a_0 a_1 = 0 \]  

(5.17c)

\[ 180\alpha a_1 a_2^2 + 60\alpha^5 a_2 - 20 \alpha \beta a_0 - 10 \alpha \beta a_1 + 150\alpha^3 a_1 a_2 + 30\alpha^3 a_1^2 + 720\alpha a_2 a_0 \]  

\[ +60\alpha a_1^3 - 4\alpha \beta a_2 + 360\alpha a_1 a_2 a_0 + 720\alpha a_0^3 a_2 + 1200\alpha a_0^3 + 1800\alpha a_1 a_0^2 = 0 \]  

(5.17d)

\[ -10\alpha^5 a_1 + 16\alpha^5 a_2 + 180\alpha a_0^3 a_2 - 5\alpha \beta a_1 + 900\alpha a_0^3 + 60\alpha^3 a_0 a_1 + 180\alpha a_2 a_0^2 \]  

\[ +900\alpha a_0^2 a_2 + 120\alpha a_1 a_0^2 - 15\alpha \beta a_0 - \alpha \beta a_2 + 30\alpha^3 a_1^2 = 0 \]  

(5.17e)

\[ 360\alpha a_0^3 + \alpha^5 a_1 + 180\alpha a_1 a_0^2 + 30\alpha^3 a_0 a_1 - 6\alpha \beta a_0 - \alpha \beta a_1^2 = 0 \]  

(5.17f)

\[ -\alpha \beta a_0 + 60\alpha a_0^3 = 0 \]  

(5.17g)
With the aid of Maple, the solutions of these algebraic equations are found to be:

\[ a_0 = 0, \; a_1 = \alpha^2, \; a_2 = -\alpha^2, \; \beta = \alpha^4 \]  \hspace{1cm} (5.18a)

\[ a_0 = \frac{1}{2} \left( -\frac{1}{4} + \frac{1}{60} \sqrt{105} \right) \alpha^2, \; a_1 = \alpha^2, \; a_2 = -\alpha^2, \; \beta = \frac{1}{2} \left( -\frac{11}{4} + \frac{1}{4} \sqrt{105} \right) \alpha^4 \]  \hspace{1cm} (5.18b)

Substituting Eq. (5.18a) and Eq. (5.18b) in Eq. (5.16), we obtain two soliton solutions for Eq. (1.1) of the form:

\[ u_3 = \frac{\alpha^2}{1 + e^{(\alpha(x-a_3t))}} - \frac{\alpha^2}{(1 + e^{(\alpha(x-a_3t))})^2} = \frac{\alpha^2 (\cosh (\alpha (x + \alpha^4 t)) + \sinh (\alpha (x + \alpha^4 t)))}{(1 + \cosh (\alpha (x + \alpha^4 t)) + \sinh (\alpha (x + \alpha^4 t)))^2} \]  \hspace{1cm} (5.19a)

\[ u_4 = \frac{\alpha^2}{1 + e^{(\alpha(x-\frac{1}{2}(\frac{1}{4} + \frac{1}{4} \sqrt{105}) \alpha^4 t))}} - \frac{\alpha^2}{(1 + e^{(\alpha(x-\frac{1}{2}(\frac{1}{4} + \frac{1}{4} \sqrt{105}) \alpha^4 t))})^2} \] \hspace{1cm} (5.19b)

### 5.3 Application of rational sinh and cosh functions methods

We next substitute the rational sinh method (4.1) for \( n = 1 \) into Eq.(5.2) and Collect the coefficients of the same power of resulting hyperbolic equal to zero, the following algebraic system will be obtained:

\[ -8b_0 \mu^4 a_1 + 8a_1^2 \mu^4 a_0 - 24b_0 \mu^4 a_3 + 60 \mu^2 a_2^2 a_0 + 24a_1^4 \mu^4 a_0 + 60a_0 - 60 \mu^2 a_0 b_0 a_1 = 0 \]  \hspace{1cm} (5.20a)

\[ 180a_0 b_0 - 20a_1^2 \mu^4 a_0 - c b_0 - a_1 \mu^4 a_0 - 4c a_0 a_1 - 30 \mu^2 a_0^2 a_1 + b_0 \mu^4 + 120 a_0^2 a_1 \]

\[ + 60 \mu^2 a_3^2 a_0^2 + 20b_0 \mu^4 a_1^2 + 30 \mu^2 a_0 b_0 - 60 \mu^2 b_0^2 a_1 = 0 \]  \hspace{1cm} (5.20b)

\[ -20b_0 \mu^4 a_1^2 - 60 \mu^2 b_0^2 a_1^2 + 180a_0 b_0^2 + 11a_1^2 \mu^4 a_0 + 30 \mu^2 b_0^2 + 60 \mu^2 b_0 a_1 a_0 \]

\[ + 20a_1^4 \mu^4 a_0 - 4c b_0 a_1 + 60a_3^2 a_1 - 11b_0 \mu^4 a_1 + 360a_0^2 b_0 a_1 - 6ca_1 a_0 - 30 \mu^2 a_0 b_0 a_1 = 0 \]  \hspace{1cm} (5.20c)

\[ 60b_0^2 - 11a_1^2 \mu^4 a_0 + 360a_0 b_0 a_1^2 + 11b_0 \mu^4 a_1 + 30 \mu^2 a_1^2 a_0 - 30 \mu^2 b_0 a_1 a_0 \]

\[ -4c a_0 a_1 - 6c b_0 a_1^2 + 180a_0^2 b_0 a_1^2 = 0 \]  \hspace{1cm} (5.20d)

\[ 120b_0 a_1 - b_0 \mu^4 a_1^2 - c a_0 a_1 + 30 \mu^2 b_0 a_1 a_0 - 30 \mu^2 b_0 a_1^2 + a_1^4 \mu^4 a_0 + 180a_0 b_0 a_1^2 \]

\[ -4c b_0 a_1^2 = 060b_0 b_0^2 - c b_0 a_1^2 = 0 \]  \hspace{1cm} (5.20e)

Solving the above system, we the following results will be obtained:

\[ a_0 = \frac{1}{2} \mu^2, \; a_1 = I, \; b_0 = 0, \; c = \mu^4 \]  \hspace{1cm} (5.21a)

\[ a_0 = \left( \frac{3}{8} - \frac{1}{120} \sqrt{105} \right) \mu^2, a_1 = I, b_0 = \frac{1}{2} \mu^2 \left( -\frac{1}{4} - \frac{1}{60} \sqrt{105} \right), c = -15 \mu^4 \left( \frac{3}{8} - \frac{1}{120} \sqrt{105} \right) + 7 \mu^4 \]  \hspace{1cm} (5.21b)
Substituting Eq. (5.21a) and Eq. (5.21b) in Eq. (4.2), we obtain the complex solutions of Eq. (1.1):

\[ u_5 = \frac{\mu^2}{2(1 + I \sinh (\mu (x - \mu^4 t)))} \]  
\[ u_6 = \frac{(\frac{3}{8} - \frac{1}{30} \sqrt{105}) \mu^2}{1 + I \sinh (\mu (x - (-15\mu^4 (\frac{3}{8} - \frac{1}{30} \sqrt{105}) + 7\mu^4) t))} \]

If we choose \( n = 2 \), and consider the rational sinh method solution form of (4.1) and insert them into Eq. (5.2), and Collect the coefficients of the same power of resulting hyperbolic equal to zero, the following algebraic system will be resulted:

\[ 24a_1^2 \mu^4 a_0 - 8a_1 \mu^4 a_0 + 8b_0 \mu^4 + 60a_0^3 - 24b_0 \mu^4 a_1 - 60\mu^2 a_1^2 - ca_0 + 60\mu^2 a_0 b_0 = 0 \]  
\[ -4ca_0 a_1 + 180a_0^2 b_0 - 200b_0 \mu^4 a_1 - 120\mu^2 a_1 a_0^2 + 120a_0^3 a_1 - 16a_1 \mu^4 a_0 + 120\mu^2 a_0 b_0 \]

\[ + 200a_1^2 \mu^4 a_0 + 160b_0 \mu^4 a_1^2 - 240a_1^3 \mu^4 a_0 - cb_0 + 120\mu^2 a_1^2 a_0^2 - 180\mu^2 a_0 b_0 a_1 + 60\mu^2 b_0^2 = 0 \]

\[ -176b_0 \mu^4 a_1 - 440a_1^2 \mu^4 a_1^2 - 120b_0 \mu^4 a_1^2 + 180a_0 b_0^2 - 60\mu^2 b_0 a_1 a_0 + 180\mu^2 a_1^3 a_0^2 - 4cb_0 a_1 \]

\[ -6ca_0 a_1^2 + 60a_0^3 a_1^2 + 176a_1^3 \mu^4 a_0 + 440b_0 \mu^4 a_1^2 + 120a_1^4 \mu^4 a_0 + 360a_0^3 b_0 a_1 + 120\mu^2 b_0^2 - 120\mu^2 a_0^2 a_1 = 0 \]

\[ -6cb_0 a_1^2 + 176b_0 \mu^4 a_1^2 + 120\mu^2 a_1^3 a_0^2 - 120b_0 \mu^4 a_1^2 a_0 - 180\mu^2 b_0 a_1 a_0^2 - 4ca_0 a_1^3 + 180\mu^2 a_1^3 a_0 \]

\[ + 360a_0 b_0 a_1^2 - 120b_0 \mu^4 a_1^3 + 60a_0^3 b_0 a_1^2 - 176a_1^3 \mu^4 a_0 + 120a_1^4 \mu^4 a_0 = 0 \]

\[ 120\mu^2 b_0 a_1^3 a_0 - 16b_0 \mu^4 a_1^3 + 120b_0^3 a_1 + 180a_0 b_0 a_1^2 - 120\mu^2 b_0 a_1^3 a_1^2 - 4cb_0 a_1^3 - ca_0 a_1^4 + 16a_1^4 \mu^4 a_0 = 0 \]

\[ 60a_0^3 a_1^2 - cb_0 a_1^4 = 0 \]

With the aid of Maple, the solutions of these algebraic equations are found to be:

\[ a_0 = \mu^2, a_1 = 1, b_0 = 0, c = 16\mu^4 \]  
\[ a_0 = \left( \frac{1}{2} - \frac{1}{30} \sqrt{105} \right) \mu^2, a_1 = 1, b_0 = \left( \frac{1}{2} - \frac{1}{30} \sqrt{105} \right) \mu^2 - \mu^2, c = -60\mu^4 \left( \frac{1}{2} - \frac{1}{30} \sqrt{105} \right) + 52\mu^4 \]

Substituting Eq. (5.24a) and Eq. (5.24b) in Eq. (4.2) along with \( n = 2 \), we obtain another solitons solutions of Eq. (1.1):

\[ u_7 = \frac{\mu^2}{1 + \sinh^2 (\mu (x - 16\mu^4 t)))} = \mu^2 \sec h^2 (\mu (x - 16\mu^4 t))) \]  
\[ u_8 = \frac{(\frac{1}{2} - \frac{1}{30} \sqrt{105}) \mu^2}{1 + \sinh^2 (\mu (x - (-60\mu^4 (\frac{1}{2} - \frac{1}{30} \sqrt{105}) + 52\mu^4) t)))} \]

\[ + \left( \left( \frac{1}{2} - \frac{1}{30} \sqrt{105} \right) \mu^2 - \mu^2 \sinh^2 (\mu (x - (-60\mu^4 (\frac{1}{2} - \frac{1}{30} \sqrt{105}) + 52\mu^4) t))) \right) \]

\[ = \left( \frac{1}{2} - \frac{1}{30} \sqrt{105} \right) \mu^2 - \mu^2 + \mu^2 \sec h^2 \left( \mu \left( x - (-60\mu^4 (\frac{1}{2} - \frac{1}{30} \sqrt{105}) + 52\mu^4) t) \right) \right) \]
6 Conclusion

In summary, we have applied the sin-cosine method along with rational exponential function and rational sinh method to obtain travelling wave solution for the Caudrey-Dodd-Gibbon (CDG) equation. Solitary wave solutions, periodic solutions, and complex solutions were formally derived. In fact, the present methods are readily applicable to a large variety of such nonlinear equations and it was found that the employed methods provide different solutions. It is showing that the sin-cosine method and the rational exponential function method are powerful and straightforward solution method to find closed-form analytical expressions for travelling waves of nonlinear evolution equations.

References


