Exact solutions of Kupershmidt equation by the \( (G'/G) \)-expansion method

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Abstract:
The \((G'/G)\)-expansion method can be used to construct exact travelling wave of nonlinear evolution equations. In this paper, we look for exact solutions of Kupershmidt equation by the \((G'/G)\)-expansion method.

Keywords: \((G'/G)\)-expansion method; Kupershmidt equation.

1 Introduction

Phenomena in physics and other fields are often described by nonlinear evolution equation. When we want to understand the physical mechanism of phenomena in nature, described by nonlinear evolution equations, exact solutions for the nonlinear evolution equations have to be explored. For example, the wave phenomena observed in fluid dynamics [1, 2], plasma and elastic media [3, 4] and optical fibers [5, 6], etc. Thus, the methods for deriving exact solutions for the governing equations have to be developed. Recently, many powerful methods have been established and improved. Among these methods, we cite the homogeneous balance method [7, 8], the tanh-function method [9], the extended tanh-function method [10], the jacobi elliptic function expansion method [11, 12], the auxiliary equation method [13] and so on.

Very recently, Wang et al. [14] introduced a new method called the \((G'/G)\)-expansion method to look for travelling wave solutions of nonlinear evolution equations. The \((G'/G)\)-expansion method is based on the assumptions that the travelling wave solutions can be expressed by a polynomial in \((G'/G)\), and that \(G = G(\xi)\) satisfies a second order linear ordinary differential equation (ODE). The degree of the polynomial can be determined by considering the homogeneous balance between the highest order derivative and nonlinear terms appearing in the given nonlinear evolution equations. The coefficients of the polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the method. By using the \((G'/G)\)-expansion method, Wang et al. successfully obtain more travelling wave solutions of four nonlinear evolution equations.

Lately, work has been done on the extensions of the \((G'/G)\)-expansion method. For example, in [15], the method was improved to deal with the mKdV equation with variable coefficients. In [16], the
method was improved to find more types of nontravelling wave and coefficient function solutions. The aim of this paper is to find exact solutions of Kupershmidt equation in the form

\[ u_t = u_{xxxxx} + 10uu_{xxx} + 25u_xu_x + 20u^2u_x \]

by the \((G'/G')\)-expansion method, where \(u = u(x,t)\).

2 Description of the \((G'/G')\)-expansion method

Suppose that a nonlinear equation, say in two independent variables \(x\) and \(t\), is given by

\[ P(u, u_t, u_x, u, u_{xx}, u_{xxx}, ... ) = 0, \quad (2.1) \]

where \(u = u(x,t)\) is an unknown function, \(P\) is a polynomial in \(u = u(x,t)\) and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the \((G'/G')\)-expansion method.

**Step 1.** Combining the independent variables \(x\) and \(t\) into one variable \(\xi = x - vt\), we suppose that

\[ u(x,t) = u(\xi), \quad \xi = x - vt, \quad (2.2) \]

the travelling wave variable (2) permits us reducing Eq.(1) to an ODE for \(u = u(\xi)\)

\[ P(u, -vu', v^2u'', -vu''', u''', ... ) = 0, \quad (2.3) \]

**Step 2.** Suppose that the solution of ODE (3) can be expressed by a polynomial in \((G'/G')\) as follows:

\[ u(\xi) = \alpha_m(G'/G)^m + ..., \quad (2.4) \]

where \(G = G(\xi)\) satisfies the second order LODE in the form

\[ G'' + \lambda G' + \mu G = 0, \quad (2.5) \]

\(\alpha_m, ..., \lambda\) and \(\mu\) are constants to be determined later, \(\alpha_m \neq 0\), the unwritten part in (4) is also a polynomial in \((G'/G')\), but the degree of which is generally equal to or less than \(m - 1\). The positive integer \(m\) can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (3).

**Step 3.** By substituting (4) into (3) and using second order LODE (5), collecting all terms with the same order of \((G'/G')\) together, the left-hand side of Eq.(3) is converted into another polynomial in \((G'/G')\). Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for \(\alpha_m, ..., v, \lambda\) and \(\mu\).

**Step 4.** Assuming that the constants \(\alpha_m, ..., v, \lambda\) and \(\mu\) can be obtained by solving the algebraic equations in Step 3, since the general solutions of the second order LODE (5) have been well known for us, then substituting \(\alpha_m, ..., v\) and the general solutions of Eq.(5) into (4) we have more travelling wave solutions of the nonlinear evolution equation (2).

3 Kupershmidt equation

In this section we study the Kupershmidt equation

\[ u_t = u_{xxxxx} + 10uu_{xxx} + 25u_xu_x + 20u^2u_x \quad (3.1) \]

Using the wave variable \(u(x,t) = u(\xi), \quad \xi = x - vt\), carries the PDE (6) into the ODE

\[ -v \frac{\partial u(\xi)}{\partial \xi} - \frac{\partial^3 u(\xi)}{\partial \xi^3} - 10 \frac{\partial^3 u(\xi)}{\partial \xi^3} u(\xi) - 25 \frac{\partial^2 u(\xi)}{\partial \xi^2} \frac{\partial u(\xi)}{\partial \xi} - 20u^2(\xi) \frac{\partial u(\xi)}{\partial \xi} = 0. \quad (3.2) \]
Suppose that the solution of ODE (7) can be expressed by a polynomial in \( \left( \frac{G'}{G} \right) \) as follows:

\[
    u(\xi) = \alpha_m \left( \frac{G'}{G} \right)^m + ..., \tag{3.3}
\]

where \( G = G(\xi) \) satisfies the second order LODE in the form

\[
    G'' + \lambda G' + \mu G = 0. \tag{3.4}
\]

By using (8) and (9) it is easily derived that

\[
    u^2 = \alpha_m^2 \left( \frac{G'}{G} \right)^{2m} + ..., \tag{3.5}
\]

\[
    u' = -m \alpha_m \left( \frac{G'}{G} \right)^{m+1} + ..., \tag{3.6}
\]

\[
    u'' = m(m+1) \alpha_m \left( \frac{G'}{G} \right)^{m+2} + ..., \tag{3.7}
\]

\[
    \vdots
\]

\[
    u^{(5)} = -m(m+1)(m+2)(m+3)(m+4) \alpha_m \left( \frac{G'}{G} \right)^{m+5} + ..., \tag{3.8}
\]

\[
    u^2 u' = -m \alpha_m^3 \left( \frac{G'}{G} \right)^{3m+1} + .... \tag{3.9}
\]

Considering the homogeneous balance between \( u^2 u' \) and \( u^5 \) in Eq.(7), based on (11) we required that \( 3m + 1 = m + 5 \implies m = 2 \), so we can write (8) as

\[
    u(\xi) = \alpha_2 \left( \frac{G'}{G} \right)^2 + \alpha_1 \left( \frac{G'}{G} \right) + \alpha_0, \quad \alpha_2 \neq 0, \tag{3.10}
\]

and therefore

\[
    u^{(5)} = -720 \alpha_2 \left( \frac{G'}{G} \right)^7 + ..., \tag{3.11}
\]

\[
    u^2 u' = -2 \alpha_2^3 \left( \frac{G'}{G} \right)^7 + .... \tag{3.12}
\]

By substituting (10) - (14) into Eq.(7) and collecting all terms with the same power of \( \left( \frac{G'}{G} \right) \) together, the left-hand side of Eq.(7) is converted into another polynomial in \( \left( \frac{G'}{G} \right) \). Equating each coefficient of this polynomial to zero, yields a set of simultaneous algebraic equations for \( \alpha_2, \alpha_1, \alpha_0, v, \mu \) and \( \lambda \) as follows:

\[
    0 : 25 \lambda \mu^2 \alpha_1^2 + 22 \lambda^2 \mu^2 \alpha_1 + 16 \mu^3 \alpha_1 + v \mu \alpha_1 + 50 \mu^3 \alpha_1 \alpha_2 + 20 \mu^2 \alpha_2 = 0,
\]

\[
    +20 \mu^2 \alpha_0 \alpha_1 + 10 \lambda^2 \mu \alpha_0 \alpha_1 + \lambda^4 \alpha_1 + 60 \lambda \mu^2 \alpha_0 \alpha_2 + 120 \lambda \mu^3 \alpha_2 + 30 \lambda^3 \mu \alpha_2 = 0, \tag{3.13}
\]

\[
    1 : 60 \lambda^2 \mu \alpha_1^2 + 70 \mu^2 \alpha_1^2 + 40 \mu \alpha_0 \alpha_1^2 + 136 \lambda \mu \alpha_1 \alpha_2 + 52 \lambda^3 \alpha_1 + 80 \lambda \mu \alpha_0 \alpha_1
\]

\[
    +10 \lambda \mu \alpha_0 \alpha_1 + 310 \lambda \mu \alpha_1 \alpha_2 + 20 \lambda \mu \alpha_1 + \lambda^2 \alpha_2 + v \lambda \mu \alpha_1 + 584 \lambda^2 \mu \alpha_2 + 10 \mu^3 \alpha_2 + 62 \lambda \mu \alpha_2
\]

\[
    +2 \mu \mu \alpha_2 + 118 \lambda \mu \alpha_0 \alpha_2 - 82 \mu \alpha_0 \alpha_2 + 262 \mu \alpha \alpha_2 + 40 \mu \lambda \alpha \alpha_2 + 100 \mu \alpha_2 = 0,
\]

\[
    2 : 20 \alpha_1^3 + 40 \lambda \alpha_0 \alpha_1^2 + 230 \lambda \mu \alpha_1 \alpha_2 + 10 \lambda \alpha_1 \alpha_2 + 25 \lambda \alpha_2 + \nu \alpha_1 + 10 \lambda \alpha_1
\]

\[
    +10 \mu \alpha_1 \alpha_2 + 12 \mu \alpha_1 + 20 \lambda \mu \alpha_2 + 6 \lambda \alpha_2 + 530 \mu \alpha_2 + 20 \alpha_0 \alpha_2 + 120 \mu \alpha_0 \alpha_2
\]

\[
    +500 \lambda \alpha_1 \alpha_2 + 80 \mu \alpha_0 \alpha_1 + 70 \lambda \alpha_0 \alpha_1 + 90 \lambda \mu \alpha_1 + 15 \lambda \alpha_1 - 924 \lambda \mu \alpha_2 + 884 \lambda \mu \alpha_2
\]

\[
    +32 \lambda \alpha_2 + 80 \lambda \alpha_0 \alpha_2 + 520 \lambda \mu \alpha_0 \alpha_2 + 460 \lambda \mu \alpha_2 + 2636 \lambda \mu \alpha_2 + 2 \nu \alpha_2 + 40 \alpha_0 \alpha_2 = 0,
\]
Substituting the general solutions of Eq.(9) into (16) we have three types of travelling wave solutions

When

\[\xi = \frac{1}{2}\left(\frac{1}{4}\lambda^2 - 4\mu\right)\times \left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right)^2 + \frac{3}{8}\lambda^2 + \alpha_0,\]

\[\xi = x + \left(\frac{9}{4\pi}\lambda^4 + \frac{3}{4}a_0\lambda^2 + \alpha_0^2\right)t, \quad \mu = -\frac{1}{8}\lambda^2 - \alpha_0.\]

Solving the algebraic equations above with aid Maple, yields

**Case A:**

\[\mu = -\frac{1}{8}\lambda^2 - \alpha_0, \quad v = -\frac{9}{64}\lambda^4 - \frac{3}{4}a_0\lambda^2 - \alpha_0^2, \quad \alpha_1 = -\frac{3}{2}\lambda, \quad \alpha_2 = -\frac{3}{2}.\]  

(3.10)

\[\alpha_0 \text{ and } \lambda \text{ are arbitrary constants.}

By using (15), expression (12) can be written as

\[u(\xi) = -\frac{3}{2}(\frac{G'}{G})^2 - \frac{3}{2}\lambda(\frac{G'}{G}) + \alpha_0,\]

(3.11)

where \(\xi = x + \left(\frac{9}{4\pi}\lambda^4 + \frac{3}{4}a_0\lambda^2 + \alpha_0^2\right)t, \quad \mu = -\frac{1}{8}\lambda^2 - \alpha_0.\) Eq.(16) is the formula of a solution of Eq.(7).

Substituting the general solutions of Eq.(9) into (16) we have three types of travelling wave solutions of the Kupershmidt equation as follows:

When \(\lambda^2 - 4\mu > 0,\)

\[u_1(\xi) = \frac{3}{8}(\lambda^2 - 4\mu)\times \left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right)^2 + \frac{3}{8}\lambda^2 + \alpha_0,\]

\[\xi = x + \left(\frac{9}{4\pi}\lambda^4 + \frac{3}{4}a_0\lambda^2 + \alpha_0^2\right)t, \quad \mu = -\frac{1}{8}\lambda^2 - \alpha_0.\]

If \(C_1 \text{ and } C_2 \text{ are taken as special values, the various known results in the literature can be rediscovered, for instance, if } C_1 > 0, \quad C_2 = 0, \quad \text{and } C_1^2 > C_2^2, \text{then } u_1 = u_1(\xi) \text{ can be written as:}

\[u_1(\xi) = \frac{3}{8}(\lambda^2 - 4\mu)\text{sech}^2\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi\right) + \frac{3}{2}\mu + \alpha_0,\]

where \(\xi = x + \left(\frac{9}{4\pi}\lambda^4 + \frac{3}{4}a_0\lambda^2 + \alpha_0^2\right)t, \quad \mu = -\frac{1}{8}\lambda^2 - \alpha_0.\)

When \(\lambda^2 - 4\mu < 0,\)

\[u_2(\xi) = -\frac{3}{8}(4\mu - \lambda^2)\times \left(-\frac{C_1\sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2\cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2\sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}\right)^2 + \frac{3}{8}\lambda^2 + \alpha_0,\]

\[\xi = x + \left(\frac{9}{4\pi}\lambda^4 + \frac{3}{4}a_0\lambda^2 + \alpha_0^2\right)t, \quad \mu = -\frac{1}{8}\lambda^2 - \alpha_0.\]

When \(\lambda^2 - 4\mu = 0,\)

\[u_3(\xi) = -\frac{3C_2^2}{2(C_1 + C_2)^2} + \frac{3}{8}\lambda^2 + \alpha_0,\]
\[ \xi = x + \left( \frac{99}{4} \lambda^4 + \frac{33}{4} \alpha_0 \lambda^2 + \alpha_0^2 \right) t, \]  
\[ C_1 \text{ and } C_2 \text{ are arbitrary constants.} \]

**Case B:**

\[ \mu = - \frac{1}{8} \lambda^2 - \frac{1}{8} \alpha_0, \quad v = - \frac{99}{4} \lambda^4 - \frac{33}{2} \alpha_0 \lambda^2 - \frac{11}{4} \alpha_0^2, \quad \alpha_1 = -12 \lambda, \quad \alpha_2 = -12, \quad (3.12) \]

\( \alpha_0 \) and \( \lambda \) are arbitrary constants.

By using (17), expression (12) can be written as

\[ u(\xi) = -12 \left( \frac{G'}{G} \right)^2 - 12 \lambda \left( \frac{G'}{G} \right) + \alpha_0, \quad (3.13) \]

where \( \xi = x + \left( \frac{99}{4} \lambda^4 + \frac{33}{2} \alpha_0 \lambda^2 + \frac{11}{4} \alpha_0^2 \right) t, \quad \mu = - \frac{1}{8} \lambda^2 - \frac{1}{8} \alpha_0. \) Eq.(18) is the formula of a solution of Eq.(7).

Substituting the general solutions of Eq.(9) into (18) we have other three types of travelling wave solutions of the Kupershmidt equation as follows:

When \( \lambda^2 - 4 \mu > 0, \)

\[ u_4(\xi) = -3(\lambda^2 - 4 \mu) \times \left( \frac{C_1 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4 \mu} \xi + C_2 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4 \mu} \xi}{C_1 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4 \mu} \xi + C_2 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4 \mu} \xi} \right)^2 + 3 \lambda^2 + \alpha_0, \]

\( \xi = x + \left( \frac{99}{4} \lambda^4 + \frac{33}{2} \alpha_0 \lambda^2 + \frac{11}{4} \alpha_0^2 \right) t, \quad \mu = - \frac{1}{8} \lambda^2 - \frac{1}{8} \alpha_0. \)

When \( \lambda^2 - 4 \mu < 0, \)

\[ u_5(\xi) = -3(4 \mu - \lambda^2) \times \left( -\frac{C_1 \sin \frac{1}{2} \sqrt{4 \mu - \lambda^2} \xi + C_2 \cos \frac{1}{2} \sqrt{4 \mu - \lambda^2} \xi}{C_1 \cos \frac{1}{2} \sqrt{4 \mu - \lambda^2} \xi + C_2 \sin \frac{1}{2} \sqrt{4 \mu - \lambda^2} \xi} \right)^2 + 3 \lambda^2 + \alpha_0, \]

\( \xi = x + \left( \frac{99}{4} \lambda^4 + \frac{33}{2} \alpha_0 \lambda^2 + \frac{11}{4} \alpha_0^2 \right) t, \quad \mu = - \frac{1}{8} \lambda^2 - \frac{1}{8} \alpha_0. \)

When \( \lambda^2 - 4 \mu = 0, \)

\[ u_6(\xi) = - \frac{12 C_2^2}{(C_1 + C_2 \xi)^2} + 3 \lambda^2 + \alpha_0, \]

\( \xi = x + \left( \frac{99}{4} \lambda^4 + \frac{33}{2} \alpha_0 \lambda^2 + \frac{11}{4} \alpha_0^2 \right) t, \quad C_1 \text{ and } C_2 \text{ are arbitrary constants.} \)

**References**


