Index of cordiality for complete graphs and cycle

V. J. Kaneria\textsuperscript{1}, S. K. Vaidya\textsuperscript{2}

\textit{Department of Mathematics, Saurashtra University, RAJKOT - 360005.}
\textit{Email: }\textsuperscript{1}kaneria_vinodray_j@yahoo.co.in, \textsuperscript{2}samirkvaidya@yahoo.co.in

\textbf{Abstract:}
A new concept index of cordiality is introduced and index of cordiality is investigated for complete graph, cycle and star of complete graph.

\textbf{Key words}: Index of Cordiality; Complete Graph; Cycle; Star of a Graph.

1 Introduction

We begin with simple, finite and undirected graph $G = (V(G), E(G))$ with $n$ vertices. In the present investigations $K_n$ denote the complete graph on $n$ vertices, $C_n$ denote cycle with $n$ vertices and $G^{(i)}$ denote the $i^{th}$ copy of a graph $G$. For all other terminology and notations we follow Harary [1]. We will give brief summary of definitions which are useful for the present work.

\textbf{Definition 1.1.} \textit{If the vertices of the graph $G$ are assigned values subject to certain conditions then it is known as graph labeling.}

For detail survey on graph labeling one can refer to Gallian [2]. Vast amount of literature is available on different types of graph labeling. Graph labeling was introduced during 1960. At present couple of dozens labeling techniques have been found and more than thousand research papers are available.

Most interesting labeling have following three important ingredients.

- a set of numbers from which vertex labels are chosen;
- a rule that assigns a value to each edge;
- a condition that these values must satisfy.

The present discussion is in the context of one such labeling known as cordial labeling which is defined as follows.

\textbf{Definition 1.2.} \textit{Let $G$ be a graph. A mapping $f : V(G) \rightarrow \{0, 1\}$ is called binary vertex labeling of $G$ and $f(v)$ is called label of the vertex $v$ of the graph $G$ under $f$.}

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is given by $f^*(uv = e) = (f(u) + f(v)) \pmod 2$. Let $v_f(0)$, $v_f(1)$ be the number of vertices of $G$ having labels 0 and 1 respectively under $f$ and let $e_f(0)$, $e_f(1)$ be the number of edges of $G$ having labels 0 and 1 respectively under $f^*$.

\textbf{Definition 1.3.} \textit{A binary vertex labeling of a graph $G$ is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$, $|e_f(0) - e_f(1)| \leq 1$. A graph $G$ is called cordial graph if it admits cordial labeling.}
In a seminal paper Cahit [3] introduced cordial graphs and reported several results on cordiality of graphs and in [4] he proved that $K_n$ is cordial if and only if $n \leq 3$. After this many researchers have studied cordial graphs. e.g. Ho et al. [5] proved that unicyclic graph is cordial unless it is $C_{4k+2}$. Andar et al. [6] have discussed cordiality of multiple shells. Yousef [8] has discussed cordiality of union and sum of two graphs. Vaidya et al. [9, 10, 11, 12] have also discussed cordiality of various graphs.

This paper is aimed to discuss cordiality of graphs in different context. We consider the disjoint union of $k$ copies of any graph $G$ and try to find out minimum $k$ for which the disjoint union is cordial. In this context we introduce the concept of index of cordiality of a graph as follows.

**Definition 1.4.** The index of cordiality for $G$ is $n$ if union of $n$ copies of $G$ is a cordial, but union of less than $n$ copies of $G$ is not cordial.

From the above definition it is obvious that for any cordial graph $G$, the index of cordiality is precisely $1$. In the immediate section we focus over the index of cordiality for some graphs.

**Remark:** Andar et al. [7] have defined index of cordiality($t(G)$) in a different context. Interested readers are referred to the cited reference.

### 2 On Index of Cordiality of $K_n$

**Theorem 2.1.** The index of cordiality of $K_n$ is $2$, where $n = t^2$, for some $t \in N \setminus \{1\}$.

**Proof.** Let $f$ be a binary labeling of $K_n \cup K_n$. Consider the first copy of $K_n$ for which $f^{-1}(0) = v_f(0) = l$, so that $f^{-1}(1) = v_f(1) = n - l$ (Where $n \geq 4$). In this case $e_f(0) = lC_2 + n-lC_2$ and $e_f(1) = (n-l)$. For the second copy of $K_n$ if $f^{-1}(0) = v_f(0) = n - l$ and $f^{-1}(1) = v_f(1) = l$, then obviously $e_f(0)$ and $e_f(1)$ are same as in the first copy of $K_n$. Thus the difference between $e_f(0)$ and $e_f(1)$ in $K_n \cup K_n$ is precisely $2 | 2l^2 - 2nl + n^2 |$. This is equal to 1 is impossible as $l$ is whole number and it is 0 iff $l = \frac{n+\sqrt{n}}{2}$. This is possible only when $n = t^2$, for some $t \in N \setminus \{1\}$. Thus we have $v_f(0) = v_f(1)$ and $e_f(0) = e_f(1)$ in $K_n \cup K_n$.

Thus we proved that $K_n \cup K_n$ is cordial and as stated earlier only one copy of $K_n$ is not cordial for $n \geq 4$. This implie the index of cordiality of $K_n$ $(n \geq 4)$ is 2, where $n = t^2$, for some $t \in N \setminus \{1\}$. \[\Box\]

**Illustration 2.1.** For better understanding of the above defined labeling pattern consider $K_4 \cup K_4$ as shown in Figure–1 and its conditions for cordial labeling in Table–1.

![Figure-1: $K_4 \cup K_4$ and its cordial labeling](image)

<table>
<thead>
<tr>
<th>$\mathcal{V}(0)$</th>
<th>$\mathcal{V}(1)$</th>
<th>$\mathcal{E}(0)$</th>
<th>$\mathcal{E}(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

**Table-1: Vertex and edge conditions**

**Theorem 2.2.** The index of cordiality of $C_n$ is $2$, where $n = 4k + 2$, for some $k \in N$.

**Proof.** As proved by Cahit [3] $C_n$ is not cordial for $n \equiv 2$ (mod 4). So it is enough to prove that $C_n \cup C_n$ is cordial graph.

Let $u_{1,1}, u_{1,2}, \ldots, u_{1,\alpha}$ be vertices of $\bigcup_{i=1}^{\alpha} C_n^{(i)}$, $i = 1,2$, where $C_n^{(i)} = C_n$, $\forall i = 1,2$. We define labeling function $f : \bigcup_{i=1}^{\alpha} C_n^{(i)} \rightarrow \{0,1\}$ as follows :

$f(u_{1,i}) = 0$ if $i \equiv 0,1$ (mod 4)  
$f(u_{1,i}) = 1$ if $i \equiv 2,3$ (mod 4), $\forall 1 \leq i \leq n$  
$f(u_{2,i}) = 0$ if $i \equiv 1,2$ (mod 4)  
$f(u_{2,i}) = 1$ if $i \equiv 0,3$ (mod 4), $\forall 1 \leq i \leq n-2$, $i = n$ and $f(2_{n-1}) = 1$.

Then the graph $\bigcup_{i=1}^{\alpha} C_n^{(i)}$ has $v_f(0) = v_f(1)$ and $e_f(0) = e_f(1)$. Thus the graph $\bigcup_{i=1}^{\alpha} C_n^{(i)}$ admits cordial labeling and hence the index of cordiality of $C_n$ is 2. \[\Box\]

**Illustration 2.2.** For better understanding of the above discussed labeling pattern consider $C_6 \cup C_6$ as shown in Figure–2 and its conditions for cordial labeling in Table–2.
Theorem 2.3. The index of cordiality of $K_n$ is at most $4$, where $n = t^2 + 2$ and $t \in N$.

Proof. We will show that $\bigcup_{i=1}^{4} K_n^{(i)}$ is a cordial graph, where $n = t^2 + 2$ for some $t \in N$.

Let $t_1 = \frac{1}{2}(t^2 + t)$, $t_2 = \frac{1}{2}(t^2 - t)$. Take $v_f(0) = t_2, t_1+1, t_1+1$ and $t_2+2$, so that $v_f(1)$ is $n - t_2 = t_1+2$, $n - t_1 - 1 = t_2 + 1$, $n - t_1 - 1 = t_2 + 1$ and $n - t_2 - 2 = t_1$ for $K_n^{(1)}, K_n^{(2)}, K_n^{(3)}$ and $K_n^{(4)}$ respectively.

In this case for $K_n^{(1)}, K_n^{(2)}, K_n^{(3)}$ and $K_n^{(4)}$ is $t_2 C_2 + (t_1+2) C_2, (t_1+1) C_2 + (t_2+1) C_2, (t_1+1) C_2 + (t_2+1) C_2$ and $(t_2+2) C_2 + t_1 C_2$ respectively and $e_f(1)$ is $t_2(t_1+2), (t_1+1)(t_2+1), (t_1+1)(t_2+1)$ and $t_1(t_2+1)$ respectively. Thus $v_f(0), v_f(1), e_f(0)$ and $e_f(1)$ of each copy of $K_n$ tabulated in the following Table-3.

<table>
<thead>
<tr>
<th>Copy $K_n^{(i)}$</th>
<th>$v_f(0)$</th>
<th>$v_f(1)$</th>
<th>$e_f(0)$</th>
<th>$e_f(1)$</th>
<th>${U, U-t, U+t, U+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$t_2$</td>
<td>$t_1+2$</td>
<td>$U+t+1$</td>
<td>$U-t$</td>
<td>$U=(t^2+3t)$</td>
</tr>
<tr>
<td>2</td>
<td>$t_1+1$</td>
<td>$t_2+1$</td>
<td>$U$</td>
<td>$U+1$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$t_1+1$</td>
<td>$t_2+1$</td>
<td>$U$</td>
<td>$U+1$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$t_2+2$</td>
<td>$t_1$</td>
<td>$U-t+1$</td>
<td>$U+t$</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>$2t^2+4$</td>
<td>$2t^2+4$</td>
<td>$4U+2$</td>
<td>$4U+2$</td>
<td></td>
</tr>
</tbody>
</table>

Table-3: Vertex labeling for four copies of $K_n$

Hence the graph $\bigcup_{i=1}^{4} K_n^{(i)}$ is a cordial graph and the index of cordiality of $K_n \leq 4$, when $n = t^2 + 2$, for some $t \in N$.

Illustration 2.3. For better understanding of the above defined labeling pattern consider $\bigcup_{i=1}^{4} K_6^{(i)}$ as shown in following Figure-3 and its vertex and edge labels are according to Table-4.

Figure-3: $K_6 \cup K_6 \cup K_6 \cup K_6$ and its cordial labeling
Theorem 2.4. The index of cordiality of $K_n$ is at most 4, where $n = t^2 + s^2$ and $t, s \in \mathbb{N}$.

Proof. Take $t_1 = \frac{1}{2}(t^2 + t)$, $t_2 = \frac{1}{2}(t^2 - t)$, $s_1 = \frac{1}{2}(s^2 + s)$ and $s_2 = \frac{1}{2}(s^2 - s)$. Now for each copy of $K_n$, define $v_f(0)$, $v_f(1)$ according to following Table 5, which will produce the edge labels $e_f(0)$ and $e_f(1)$ for each copy of $K_n$ as given in same Table 5.

Illustration 2.4. For better understanding of the above defined labeling pattern consider $\bigcup_{i=1}^{4} K_5^{(i)}$ as shown in Figure 4.
3 Star of Graph and its Cordial Labeling

**Definition 3.1.** A graph obtained by replacing each vertex of star $K_{1,n}$ by a graph $G$ of $n$ vertices is called star of G and it is denoted by $G^*$ . The graph $G$ which replaced at the center of $K_{1,n}$ we call the central copy of $G$.

Above definition was introduced in [13] by Vaidya et al. and they also proved that star of Petersen graph, star of cycle $C_n$ are cordial.

**Theorem 3.2.** $K_{n}^*$ (the star of complete graph $K_{n}$) is cordial, where $n = t^2 + 2$ and $t$ is an odd integer.

**Proof.** Let $v_1, v_2, ..., v_n$ be vertices of central copy of $K_{n}^*$ and $u_{i,1}, u_{i,2}, ..., u_{i,n}$ be vertices of other copies $K_{n}^{(i)}$ in $K_{n}^*$, $i = 1, 2, ..., n$. We shall assume that the vertex $u_{i,1}$ of each copy $K_{n}^{(i)}$ is adjacent with the vertex $v_i$ of central copy $K_n$ in $K_{n}^*$. To define required labeling function $f : V(K_{n}^*) \rightarrow \{0, 1\}$, we shall use following labels. Here $n = t^2 + 2$, for some odd integer $t$. So $n$ is also an odd integer.

\[
\begin{align*}
  f(v_i) & = 0 \text{ if } 1 \leq i \leq \frac{t^2+t}{2} + 1, f(v_i) = 1 \text{ if } \frac{t^2+t}{2} + 2 \leq i \leq n \\
  f(u_{i,j}) & = 0 \text{ if } 1 \leq j \leq \frac{t^2-t}{2} + 1 \\
  & = 1 \text{ if } \frac{t^2-t}{2} + 2 \leq j \leq n, i \equiv 1 \pmod{2}, 1 \leq i \leq \frac{t^2+t}{2} + 1. \\
  f(u_{i,j}) & = 0 \text{ if } 1 \leq j \leq \frac{t^2-t}{2} + 1 \\
  & = 1 \text{ if } \frac{t^2-t}{2} + 2 \leq j \leq n, i \equiv 0 \pmod{2}, 1 \leq i \leq \frac{t^2+t}{2} + 1, \\
  f(u_{i,j}) & = 0 \text{ if } 1 \leq j \leq \frac{t^2-t}{2} + 1 \\
  & = 1 \text{ if } \frac{t^2-t}{2} + 2 \leq j \leq n, i \equiv 1 \pmod{2}, \frac{t^2+t}{2} + 2 \leq i \leq n, \\
  f(u_{i,j}) & = 0 \text{ if } 1 \leq j \leq \frac{t^2-t}{2} + 1 \\
  & = 1 \text{ if } \frac{t^2-t}{2} + 2 \leq j \leq n, i \equiv 0 \pmod{2}, \frac{t^2+t}{2} + 2 \leq i \leq n.
\end{align*}
\]

Now the vertex labels and edge labels of $K_{n}^*$ would be according to Table 7.

<table>
<thead>
<tr>
<th>Copy $K_n$</th>
<th>$f (0) \backslash f (1)$</th>
<th>$e_f(0) \backslash e_f(1)$</th>
<th>$U$</th>
<th>$U+1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Central Copy</td>
<td>$(t^2+t)/2 +1$</td>
<td>$(t^2-t)/2 +1$</td>
<td>U</td>
<td>U+1</td>
</tr>
<tr>
<td>Odd Copies</td>
<td>$(t^2-t)/2 +1$</td>
<td>$(t^2-t)/2 +1$</td>
<td>U</td>
<td>U+1</td>
</tr>
<tr>
<td>Even Copies</td>
<td>$(t^2-t)/2 +1$</td>
<td>$(t^2-t)/2 +1$</td>
<td>U</td>
<td>U+1</td>
</tr>
<tr>
<td>Outer Edges of central copy</td>
<td>—</td>
<td>—</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>$n(n+1)/2$</td>
<td>$n(n+1)/2$</td>
<td>$nU+n+U$</td>
<td>$nU+n+U+1$</td>
</tr>
</tbody>
</table>

Table-7: Vertex labeling for four copies of $K_n$.

Thus $K_{n}^*$ is a cordial graph, where $n = t^2 + 2$ and $t$ is an odd integer.

**Illustration 3.1.** Following Table-8 demonstrate the labeling pattern provided in the above Theorem-3.2 for $K_{11}^*$. Any one can easily see that $K_{11}^*$ is cordial.

<table>
<thead>
<tr>
<th>Copy $K$</th>
<th>$f(v)(0) \backslash f(v)(1)$</th>
<th>$e_f(0) \backslash e_f(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Central Copy</td>
<td>$7 \times 1 = 7$</td>
<td>$27 \times 1 = 27$</td>
</tr>
<tr>
<td>Odd Copies</td>
<td>$4 \times 6 = 24$</td>
<td>$28 \times 6 = 168$</td>
</tr>
<tr>
<td>Even Copies</td>
<td>$7 \times 5 = 35$</td>
<td>$27 \times 5 = 135$</td>
</tr>
<tr>
<td>Outer Edges of Central Copy</td>
<td>0</td>
<td>11 \times 1 = 11</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>66</strong></td>
<td><strong>335</strong></td>
</tr>
</tbody>
</table>

Table-8: Vertex and edge conditions

**Theorem 3.3.** $K_{n}^*$ (the star of complete graph $K_{n}$) is cordial, where $n = t^2$ or $t^2 - 2$ and $t \in N$.

**Proof.** To define labeling function $f : V(K_{n}^*) \rightarrow \{0, 1\}$, following four cases to be considered.

**Case-1:** $n = t^2$, for some $t \in N$ and $t$ is odd.

\[
f(v_i) = 0 \text{ if } 1 \leq i \leq \frac{t^2-t}{2} \]

\[
f(v_i) = 1 \text{ if } \frac{t^2-t}{2} + 1 \leq i \leq n.
\]
\[
= 1 \text{ if } \frac{t^2 - t}{2} + 1 \leq i \leq n \text{ and } \\
\]
f(u_i,j) = 0 \text{ if } 1 \leq j \leq \frac{t^2 - t}{2} \\
= 1 \text{ if } \frac{t^2 - t}{2} + 1 \leq j \leq n, \forall \ 1 \leq i \leq \frac{t^2 - t}{2}, \\
f(v_i) = 0 \text{ if } 1 \leq i \leq \frac{t^2 - t}{2} \\
= 1 \text{ if } \frac{t^2 - t}{2} + 1 \leq i \leq n, \forall \ j \leq \frac{t^2 - t}{2}, \\
f(u_i) = 0 \text{ if } 1 \leq j \leq \frac{t^2 - t}{2} \\
= 1 \text{ if } \frac{t^2 - t}{2} + 1 \leq j \leq n, \forall \ i \leq \frac{t^2 - t}{2}, \\
f(v_i,j) = 1 \text{ if } 1 \leq j \leq \frac{t^2 - t}{2} \\
= 0 \text{ if } \frac{t^2 - t}{2} + 1 \leq j \leq n, \ i \equiv 1 \mod 2, \forall \ t^2 - t + 1 \leq i \leq n, \\
f(u_i,j) = 0 \text{ if } 1 \leq j \leq \frac{t^2 - t}{2} \\
= 1 \text{ if } \frac{t^2 - t}{2} + 1 \leq j \leq n, \ i \equiv 0 \mod 2, \forall \ t^2 - t + 1 \leq i \leq n.
\]

Case—II : \(n = t^2\), for some \(t \in N\) and \(t\) is even.

\[
f(v_i) = 0 \text{ if } 1 \leq i \leq \frac{n}{2} \\
= 1 \text{ if } \frac{n}{2} + 1 \leq i \leq n \text{ and } \\
f(u_i,j) = 0 \text{ if } 1 \leq j \leq \frac{t^2 - t}{2} \\
= 1 \text{ if } \frac{t^2 - t}{2} + 1 \leq j \leq n, \ i \equiv 1 \mod 2, \ \forall \ 1 \leq i \leq \frac{n}{2}, \\
f(u_i) = 0 \text{ if } 1 \leq j \leq \frac{t^2 - t}{2} \\
= 1 \text{ if } \frac{t^2 - t}{2} + 1 \leq j \leq n, \ i \equiv 0 \mod 2, \ \forall \ 1 \leq i \leq \frac{n}{2}, \\
f(v_i,j) = 1 \text{ if } 1 \leq j \leq \frac{t^2 - t}{2} \\
= 0 \text{ if } \frac{t^2 - t}{2} + 1 \leq j \leq n, \ i \equiv 1 \mod 2, \ \forall \ \frac{n}{2} + 1 \leq i \leq \frac{3n}{4}, \\
f(u_i,j) = 1 \text{ if } 1 \leq j \leq \frac{t^2 - t}{2} \\
= 0 \text{ if } \frac{t^2 - t}{2} + 1 \leq j \leq n, \ i \equiv 0 \mod 2, \ \forall \ \frac{n}{2} + 1 \leq i \leq \frac{3n}{4}, \\
f(u_i) = 0 \text{ if } 1 \leq j \leq \frac{t^2 - t}{2} \\
= 1 \text{ if } \frac{t^2 - t}{2} + 1 \leq j \leq n, \ i \equiv 1 \mod 2, \ \forall \ \frac{3n}{4} + 1 \leq i \leq n, \\
f(v_i,j) = 0 \text{ if } 1 \leq j \leq \frac{t^2 - t}{2} \\
= 1 \text{ if } \frac{t^2 - t}{2} + 1 \leq j \leq n, \ i \equiv 0 \mod 2, \ \forall \ \frac{3n}{4} + 1 \leq i \leq n.
\]

Case—III : \(n = \frac{t^2}{2} - 2\), for some \(t \in N\) and \(t\) is odd.

\[
f(v_i) = 0 \text{ if } 1 \leq i \leq \frac{t^2 + t}{2} - 1 \\
= 1 \text{ if } \frac{t^2 + t}{2} \leq i \leq n \text{ and } \\
f(u_i,j) = 1 \text{ if } 1 \leq j \leq \frac{t^2 + t}{2} - 1 \\
= 0 \text{ if } \frac{t^2 + t}{2} \leq j \leq n, \ i \equiv 1 \mod 2, \ \forall \ 1 \leq i \leq \frac{t^2 + t}{2} - 1, \\
f(u_i) = 1 \text{ if } 1 \leq j \leq \frac{t^2 - t}{2} - 1 \\
= 0 \text{ if } \frac{t^2 - t}{2} \leq j \leq n, \ i \equiv 0 \mod 2, \ \forall \ 1 \leq i \leq \frac{t^2 - t}{2} - 1, \\
f(v_i,j) = 0 \text{ if } 1 \leq j \leq \frac{t^2 - t}{2} - 1 \\
= 1 \text{ if } \frac{t^2 - t}{2} \leq j \leq n, \ i \equiv 1 \mod 2, \ \forall \ \frac{t^2 + t}{2} - 1 \leq i \leq n, \\
f(u_i,j) = 0 \text{ if } 1 \leq j \leq \frac{t^2 - t}{2} - 1 \\
= 1 \text{ if } \frac{t^2 - t}{2} \leq j \leq n, \ i \equiv 0 \mod 2, \ \forall \ \frac{t^2 + t}{2} - 1 \leq i \leq n.
\]

Case—IV : \(n = \frac{t^2}{2} - 2\), for some \(t \in N\) and \(t\) is even.

\[
f(v_i) = 0 \text{ if } 1 \leq i \leq \frac{n}{2} \\
= 1 \text{ if } \frac{n}{2} + 1 \leq i \leq n \text{ and } \\
f(u_i,j) = 0 \text{ if } 1 \leq j \leq \frac{t^2 + t}{2} - 1 \\
= 1 \text{ if } \frac{t^2 + t}{2} \leq j \leq n, \ i \equiv 1 \mod 2, \ \forall \ 1 \leq i \leq \frac{3n - 2}{4}, \\
f(u_i) = 0 \text{ if } 1 \leq j \leq \frac{t^2 + t}{2} - 1 \\
= 1 \text{ if } \frac{t^2 + t}{2} \leq j \leq n, \ i \equiv 0 \mod 2, \ \forall \ 1 \leq i \leq \frac{3n - 2}{4}, \\
f(v_i,j) = 1 \text{ if } 1 \leq j \leq \frac{t^2 - t}{2} - 1 \\
= 0 \text{ if } \frac{t^2 - t}{2} \leq j \leq n, \ i \equiv 1 \mod 2, \ \forall \ \frac{3n + 2}{4} \leq i \leq n, \\
f(u_i,j) = 1 \text{ if } 1 \leq j \leq \frac{t^2 - t}{2} - 1 \\
= 0 \text{ if } \frac{t^2 - t}{2} \leq j \leq n, \ i \equiv 0 \mod 2, \ \forall \ \frac{3n + 2}{4} \leq i \leq n.
\]

The graph under consideration satisfies \(|v_f(0) - v_f(1)| \leq 1, |e_f(0) - e_f(1)| \leq 1\) in each case as
shown in Table 9. i.e. $K_n^*$ admits cordial labeling, when $n = t^2$ or $t^2 - 2$, for any integer $t \in N$.  

<table>
<thead>
<tr>
<th>Possible Cases</th>
<th>Vertex Condition</th>
<th>Edge Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = t^2$, $t$ is an odd integer</td>
<td>$v_f(0) = v_f(1)$</td>
<td>$e_f(0) = e_f(1) + 1$</td>
</tr>
<tr>
<td>$n = t^2$, $t$ is an even integer</td>
<td>$v_f(0) = v_f(1)$</td>
<td>$e_f(0) = e_f(1)$</td>
</tr>
<tr>
<td>$n = t^2 - 2$, $t$ is an odd integer</td>
<td>$v_f(0) = v_f(1)$</td>
<td>$e_f(0) = e_f(1) + 1$</td>
</tr>
<tr>
<td>$n = t^2 - 2$, $t$ is an even integer</td>
<td>$v_f(0) = v_f(1)$</td>
<td>$e_f(0) = e_f(1) + 1$</td>
</tr>
</tbody>
</table>

Table-9: Vertex and edge conditions

**Theorem 3.4.** The index of cordiality for $K_{28}$ is six.

**Proof.** First we will show that union of 5 copies of $K_{28}$ does not admits cordial labeling but union of 6 copies of $K_{28}$ is cordial graph.

The following Table-10 shows $v_f(0)$, $v_f(1)$, $e_f(0)$, $e_f(1)$ for each copy of $K_{28}$.

<table>
<thead>
<tr>
<th>Row</th>
<th>$v_f(0)$ or $v_f(1)$</th>
<th>$e_f(0)$</th>
<th>$e_f(1)$</th>
<th>$e_f(0) - e_f(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>14</td>
<td>182</td>
<td>196</td>
<td>14</td>
</tr>
<tr>
<td>b</td>
<td>15</td>
<td>13</td>
<td>183</td>
<td>12</td>
</tr>
<tr>
<td>c</td>
<td>16</td>
<td>12</td>
<td>186</td>
<td>6</td>
</tr>
<tr>
<td>d</td>
<td>17</td>
<td>11</td>
<td>191</td>
<td>-4</td>
</tr>
<tr>
<td>e</td>
<td>18</td>
<td>10</td>
<td>198</td>
<td>-18</td>
</tr>
<tr>
<td>f</td>
<td>19</td>
<td>0</td>
<td>207</td>
<td>-36</td>
</tr>
<tr>
<td>g</td>
<td>20</td>
<td>0</td>
<td>218</td>
<td>-58</td>
</tr>
<tr>
<td>h</td>
<td>28</td>
<td>0</td>
<td>378</td>
<td>-378</td>
</tr>
</tbody>
</table>

Table-10: Cordial labeling for vertices and edges of $K_{28}$

Now the difference $e_f(0) - e_f(1)$ is even for any $v_f(0)$ and $v_f(1)$. For cordial labeling of union of any number of copies of $K_{28}$ such that $e_f(0) - e_f(1)$ is equal to zero, i.e. $e_f(1) = e_f(0)$ union of any copies of $K_{28}$, which are less than or equal to 5, following vertex labels are possible. These are $2c + 3d, 3c + e, b + c + e, 3b + f, 2b + 2c + f, b + 3d, 3b + 2e, 2a + b + d + f$ and $a + b + 2d + e$, where $a, b, c, d, e, f$ are the rows of Table 10.

For these all possible of vertex labeling of union of some copies of $K_{28}$, the difference of vertex labels are shown in following Table-11.

<table>
<thead>
<tr>
<th>Sr No</th>
<th>Rows which makes $e_f(0) = e_f(1)$</th>
<th>Difference of vertex labels</th>
<th>Minimum difference of vertex labels</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2c + 3d$</td>
<td>4, 6, 6, 6, 6</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>$3c + e$</td>
<td>4, 4, 4, 8</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>$b + c + e$</td>
<td>2, 4, 8</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>$3b + f$</td>
<td>2, 2, 2, 10</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>$2b + 2c + f$</td>
<td>2, 2, 4, 4, 10</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>$b + 3d$</td>
<td>2, 6, 6, 6</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>$3b + 2e$</td>
<td>2, 2, 2, 8, 8</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>$2a + b + d + f$</td>
<td>0, 0, 2, 6, 10</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>$a + b + 2d + e$</td>
<td>0, 2, 6, 6, 8</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>$2b + 2c + 2e$</td>
<td>2, 2, 4, 4, 8, 8</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>$a + 2c + 2d + e$</td>
<td>0, 4, 4, 6, 6, 8</td>
<td>0</td>
</tr>
</tbody>
</table>

Table-11: Table shows minimum difference of vertex labels when $e_f(0) = e_f(1)$ for all possibility
Last two rows of Table—11 show that these vertex labeling of union of six copies of $K_{28}$ is cordial. Thus the union of five or less copies of $K_{28}$ can not admit any cordial labeling, while the union of six copies of $K_{28}$ is cordial. Therefore the index of cordiality of $K_{28}$ is precisely six.

**Theorem 3.5.** $K_{28}^*$, the star of complete graph $K_{28}$ is a cordial graph.

**Proof.** Let $v_1,v_2,\ldots,v_{28}$ be vertices of central copy of $K_{28}^*$ and $u_{i,1},u_{i,2},\ldots,u_{i,28}$ be vertices of other copies $K_{28}^{(i)}$ in $K_{28}^*$, $i = 1,2,\ldots,28$. We shall assume that the vertex $u_{i,1}$ of each copy $K_{28}^{(i)}$ is adjacent with the vertex $v_i$ of central copy of $K_{28}^*$.

To define required labeling function $f : V(K_{28}^*) \to \{0,1\}$, we shall use following Table—12 and vertex labels which are given below:

- $f(v_i) = 0$ if $1 \leq i \leq 14$,
- $f(v_i) = 1$ if $15 \leq i \leq 28$ and
- $f(u_{11}) = 1$, $f(u_{i,1}) = 0$, $\forall 2 \leq i \leq 28$.

<table>
<thead>
<tr>
<th>Order of copy in $K_{28}^{(i)}$</th>
<th>$v_f(0)$</th>
<th>$v_f(1)$</th>
<th>$e_f(0)$</th>
<th>$e_f(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 to 4</td>
<td>$13\times 4 = 52$</td>
<td>$15\times 4 = 60$</td>
<td>$183\times 4 = 732$</td>
<td>$195\times 4 = 780$</td>
</tr>
<tr>
<td>4 to 8</td>
<td>$15\times 4 = 60$</td>
<td>$13\times 4 = 52$</td>
<td>$183\times 4 = 732$</td>
<td>$195\times 4 = 780$</td>
</tr>
<tr>
<td>9 to 12</td>
<td>$12\times 4 = 48$</td>
<td>$16\times 4 = 64$</td>
<td>$186\times 4 = 744$</td>
<td>$192\times 4 = 768$</td>
</tr>
<tr>
<td>13 to 16</td>
<td>$16\times 4 = 64$</td>
<td>$12\times 4 = 48$</td>
<td>$186\times 4 = 744$</td>
<td>$192\times 4 = 768$</td>
</tr>
<tr>
<td>17 to 20</td>
<td>$10\times 4 = 40$</td>
<td>$18\times 4 = 72$</td>
<td>$198\times 4 = 792$</td>
<td>$180\times 4 = 720$</td>
</tr>
<tr>
<td>21 to 24</td>
<td>$18\times 4 = 72$</td>
<td>$10\times 4 = 40$</td>
<td>$198\times 4 = 792$</td>
<td>$180\times 4 = 720$</td>
</tr>
<tr>
<td>25 to 26</td>
<td>$11\times 2 = 22$</td>
<td>$17\times 2 = 34$</td>
<td>$191\times 2 = 382$</td>
<td>$187\times 2 = 374$</td>
</tr>
<tr>
<td>27 to 28</td>
<td>$17\times 2 = 34$</td>
<td>$11\times 2 = 22$</td>
<td>$191\times 2 = 382$</td>
<td>$187\times 2 = 374$</td>
</tr>
<tr>
<td>Central Copy</td>
<td>$14$</td>
<td>$14$</td>
<td>$182\times 1 = 182$</td>
<td>$196\times 1 = 196$</td>
</tr>
<tr>
<td>Other outer edges</td>
<td>$0$</td>
<td>$0$</td>
<td>$13$</td>
<td>$15$</td>
</tr>
</tbody>
</table>

Table-12: Table shows Cordial labeling for vertices and edges of $K_{28}^*$.

Thus $K_{28}^*$ satisfies the condition $v_f(0) = v_f(1), e_f(0) = e_f(1)$ according to Table—12. Hence $K_{28}^*$ is a cordial graph.

### 4 Discussion on Index of Cordiality of $K_n$ and Cordial Labeling of The Star of $K_n$

The index of cordiality of $K_n \leq 4$ when $n \leq 27$. In Theorem—3.5 we proved that index of cordiality of $K_{28}$ is six. However we proved in Theorem—3.6 that the star of $K_{28}$ is cordial. Now a natural question arise that given any integer $m \in N$, is there $n \in N$ so that the index of cordiality of $K_n \geq m$? Next question arise that is star of complete graph $K_n$ is cordial? Third question, is there any relation between index of cordiality of $K_n$ and the cordiality of $K_n^*$? We hope that all these questions have affirmative answers and accordingly we pose following three conjectures.

**Conjecture**—4.1 : Given any integer $m \in N$, there is $n \in N$ such that the index of cordiality of $K_n \geq m$.

**Conjecture**—4.2 : For any $n \in N$, $K_n^*$ (the star of complete graph $K_n$) is cordial.

**Conjecture**—4.3 : If the index of cordiality of $K_n < n$, then $K_n^*$ admits a cordial labeling.

### 5 Concluding Remarks

The study of variety of graph labeling problems is the potential area of research. In the present work cordial labeling is discussed in the context of newly introduced concept namely index of cordiality. This work rules out the impression of cordial labeling being a weak labeling. Three powerful conjectures are posed. Similar results can be obtained for different types of labeling and for other family of graphs.
References