Non-Iterative Numerical Integration Method for Singular Perturbation Problems Exhibiting Internal and Twin Boundary Layers

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Abstract:

In this paper, a non-iterative numerical integration method is developed on a uniform mesh for a class of singularly perturbed two-point boundary value problems exhibiting internal and twin boundary layers. This method is non-iterative on a small deviating argument which converts the original second order boundary value problem to the first order differential equation with the deviating argument. By applying numerical integration method on first order differential equation, tridiagonal scheme is obtained and is solved efficiently. This method is non-iterative and very easy to implement. Relative errors with \(L_2\)-norm are presented to illustrate the proposed method.

Keywords: Singular perturbation problem, Numerical integration, Trapezoidal formula, boundary layer.

1 Introduction

Singularly perturbed boundary value problems arise frequently in many areas of science and engineering such as heat transfer problem with large Peclet numbers, Navier–Stokes flows with large Reynolds numbers, chemical reactor theory, aerodynamics, reaction–diffusion process, quantum mechanics, optimal control etc. due to the variation in the width of the layer with respect to the small perturbation parameter \(\varepsilon\). Several difficulties are experienced in solving the singular perturbation problems using standard numerical methods with uniform mesh. Equations of this type typically exhibit solutions with layers; that is, the domain of the differential equation contains narrow regions where the solution derivatives are extremely large.

The numerical treatment of singularly perturbed differential equations gives major computational difficulties due to the presence of boundary and/or interior layers. A large number of special purpose methods have been developed by various authors for singularly perturbed two-point boundary value problems \([5–12]\). P. Lin, Vancouver \([14]\) considered a quasilinear singular perturbation problem with turning points. First derivatives of the exact solution are estimated and then an approximate problem is constructed which is solved by the algorithm whose accuracy is good for...
arbitrary $\varepsilon > 0$. A.M. Watts [15] considered two asymptotic forms of solution of singular perturbation problem with a turning point and a rigorous estimate is made of the difference between the exact solutions and the asymptotic forms. S. Natesan and N. Ramamujam [16] considered exponentially fitted difference schemes to solve singular perturbation problems with twin boundary layers, where the given interval is divided into four subintervals and differential equation is solved in each interval and combines the solution.

It is well-known that replacing the first derivative by central difference is not suitable, i.e., no resemblance at all exists between the solution of the differential equation and the solution of the difference equation. This difficulty can be removed by approximating the singular perturbation problem by the first order differential equation with a small deviating argument.

In this paper, a non-iterative numerical integration method is developed on a uniform mesh for a class of singularly perturbed two-point boundary value problems exhibiting internal and twin boundary layers. In section-2, we describe the method which is non-iterative on a small deviating argument which converts the original second order boundary value problem to the first order differential equation with the deviating argument. By applying this numerical integration method on first order differential equation, tridiagonal scheme is obtained and is solved efficiently. In section-2.1, we describe the method for internal layer problems and in section-2.2, we describe the method for twin layers problems. In section-3, we demonstrate the applicability of this method computationally by considering left, right boundary layers problems and internal, twin (dual) layer problems. This method is non-iterative on deviating argument and very easy to implement. Relative errors with $L_2$-norm are presented to illustrate the proposed method.

### 2 Description Of The Method

Consider singularly perturbed boundary value problems of the form

$$Ly \equiv \varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad -1 \leq x \leq 1,$$  \hspace{1cm} (2.1)

with boundary conditions

$$y(-1) = \alpha$$  \hspace{1cm} (2.2)

and

$$y(1) = \beta$$  \hspace{1cm} (2.3)

where $0 < \varepsilon << 1$ and $\alpha, \beta$ are finite constants.

Moreover, we shall assume that in $[-1, 1]$, $b(x)$ and $f(x)$ are continuous and, for simplicity, $a(x)$ is differentiable. This problem has been treated by several authors in the last years. The behaviour of the solution depends, of course, on the properties of the functions $a(x)$ and $b(x)$. There are intervals of $[-1, 1]$ where the solution vary rapidly (layers). They may be localized either at the extreme points of the interval $[-1, 1]$ (boundary layers) or near the roots $x_i$ of $a(x)$, which are called turning points (interior layers). The following table essentially taken from the above differential equation summarizes these facts.

<table>
<thead>
<tr>
<th>Condition</th>
<th>Description</th>
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| $a(x) \neq 0 \; ; \; 0 \leq x \leq 1$ | $a(x) < 0$ boundary layer at $x = 0$
| $a(x) \neq 0$ | $a(x) > 0$ boundary layer at $x = 1$
| $a(x) = 0$ | $b(x) > 0$ boundary layers at $x=0$ and $x=1$
| $a'(x_i) \neq 0, \; a(x_i) = 0$ | $b(x) < 0$ rapidly oscillatory solution$
| a'(x_i) = 0$ | $b(x)$ changes sign (turning points)$
| $a'(x_i) < 0$ | $a'(x_i) > 0$ no boundary layers,$
| $a'(x_i) > 0$ | interior layer at $x_i$
| $a'(x_i) < 0$ | $a'(x_i) > 0$ possible boundary layers,$
| $a'(x_i) < 0$ | no interior layer at $x_i$
2.1 Internal Layer Problems

Consider the singular perturbation problem with an internal layer of the underlying interval. In this case \( a(x) \) changes sign in the domain of interest. Without loss of generality, we can take \( a(0) = 0 \) and the interval to be \([-1, 1]\). The function \( a(x) \) is defined in \([-1, 1]\) as follows:

\[
a(x) = x < 0 \text{ for } -1 \leq x < 0,
\]

\[
a(x) = x = 0 \text{ for } x = 0,
\]

\[
a(x) = x > 0 \text{ for } 0 < x \leq 1,
\]

Divide the interval \([-1, 1]\) into \( N \) equal parts with mesh size \( h \), i.e., \( h = \frac{2}{N} \) and \( x_i = -1 + ih \) for \( i = 0, 1, \ldots, N \). Let us denote \( \frac{N}{2} = l \). Then, divide the interval \([-1, 1]\) into two sub intervals \([x_{i-1}, x_i]\) for \( i = 1, 2, \ldots, l-1 \); and \([x_i, x_{i+1}]\) for \( i = l+1, l+2, \ldots, N-1 \). Let \( \sqrt{\varepsilon} \) be a small positive deviating argument.

STEP-I. By using Taylor series expansion in the neighborhood of the point \( x \in [x_{i-1}, x_i] \) for \( i = 1, 2, \ldots, l-1 \), we have

\[
y(x + \sqrt{\varepsilon}) \approx y(x) + \sqrt{\varepsilon} y'(x) + \frac{\varepsilon}{2} y''(x)
\]

and consequently, (2.1) is replaced by the following first order differential equation with a small deviation argument:

\[
y'(x) = p(x)y(x + \sqrt{\varepsilon}) + q(x)y(x) + r(x), \text{ for } -1 \leq x \leq -\sqrt{\varepsilon}
\]

where \( p(x), q(x) \) and \( r(x) \) are given by

\[
p(x) = \frac{-2}{-2\sqrt{\varepsilon} + a(x)}
\]

\[
q(x) = \frac{-2}{-2\sqrt{\varepsilon} + a(x)}
\]

\[
r(x) = \frac{f'(x)}{-2\sqrt{\varepsilon} + a(x)}
\]

Integrating (2.4) in the subinterval \([x_{i-1}, x_i]\), \( i = 1, 2, \ldots, l-1 \), we get

\[
y(x_i) - y(x_{i-1}) = \int_{x_{i-1}}^{x_i} \left( p(x)y(x + \sqrt{\varepsilon}) + q(x)y(x) + r(x) \right) \, dx
\]

By making use of trapezoidal formula for evaluating the integrals approximately, we obtain

\[
y(x_i) - y(x_{i-1}) \approx \frac{h}{2} [p(x_{i-1})y(x_{i-1} + \sqrt{\varepsilon}) + p(x_i)y(x_i + \sqrt{\varepsilon}) + q(x_{i-1})y(x_{i-1}) + q(x_i)y(x_i) + r(x_{i-1}) + r(x_i)]
\]

(2.6)

Again by means of Taylor series expansion, we have

\[
y(x + \sqrt{\varepsilon}) \approx y(x) + \sqrt{\varepsilon} y'(x)
\]

and then by approximating \( y'(x) \) by linear interpolation, we get

\[
y(x_i + \sqrt{\varepsilon}) \approx y(x_i) + \sqrt{\varepsilon} \left( \frac{y(x_{i+1}) - y(x_i)}{h} \right)
\]

(2.7)

And similarly

\[
y(x_{i-1} + \sqrt{\varepsilon}) \approx y(x_{i-1}) + \sqrt{\varepsilon} \left( \frac{y(x_i) - y(x_{i-1})}{h} \right)
\]

(2.8)
Integrating (2.11) in the subinterval \([x_i, x_{i+1}]\) and consequently, equation (2.1) is replaced by the following first order differential equation with a small deviation argument:

\[
y'(x) = p(x)y(x - \sqrt{\varepsilon}) + q(x)y(x) + r(x), \text{ for } \sqrt{\varepsilon} \leq x \leq 1
\]  

where \(p(x), q(x)\) and \(r(x)\) are given by

\[
p(x) = \frac{2\sqrt{\varepsilon + a(x)}}{2 - b(x)} \quad q(x) = \frac{2\sqrt{\varepsilon + a(x)}}{2\sqrt{\varepsilon + a(x)}} \quad r(x) = \frac{2\sqrt{\varepsilon + a(x)}}{2\sqrt{\varepsilon + a(x)}}
\]

Integrating (2.11) in the subinterval \([x_i, x_{i+1}]\), \(i = l+1, l+2, \ldots, N-1\), we get

\[
y(x_{i+1}) - y(x_i) = \int_{x_i}^{x_{i+1}} \left( p(x)y(x - \sqrt{\varepsilon}) + q(x)y(x) + r(x) \right) dx
\]

By making use of trapezoidal formula for evaluating the integrals approximately, we obtain

\[
y(x_{i+1}) - y(x_i) = \frac{h}{2} \left[ p(x_{i+1})y(x_{i+1} - \sqrt{\varepsilon}) + p(x_i)y(x_i - \sqrt{\varepsilon}) + q(x_{i+1})y(x_{i+1}) + q(x_i)y(x_i) + r(x_{i+1}) + r(x_i) \right]
\]  

Again by means of Taylor series expansion, we have

\[
y \left( x - \sqrt{\varepsilon} \right) \approx y(x) - \sqrt{\varepsilon}y'(x)
\]

And then by approximating \(y'(x)\) by linear interpolation, we get

\[
y(x_i - \sqrt{\varepsilon}) \approx y(x_i) - \sqrt{\varepsilon} \left( \frac{y(x_{i+1}) - y(x_{i-1})}{h} \right) = \left( 1 - \frac{\sqrt{\varepsilon}}{h} \right) y(x_i) + \frac{\sqrt{\varepsilon}}{h} y(x_{i-1})
\]  

And similarly

\[
y(x_{i+1} - \sqrt{\varepsilon}) \approx y(x_{i+1}) - \sqrt{\varepsilon} \left( \frac{y(x_{i+1}) - y(x_i)}{h} \right) = \left( 1 - \frac{\sqrt{\varepsilon}}{h} \right) y(x_{i+1}) + \frac{\sqrt{\varepsilon}}{h} y(x_i)
\]
Substituting equations (2.13) and (2.14) in (2.12) and rearranging, we get

\[ y(x_{i+1}) - y(x_i) = \frac{h}{2} (1 - \frac{\sqrt{\varepsilon}}{\varepsilon}) p(x_{i+1}) y(x_{i+1}) + \frac{\sqrt{\varepsilon}}{\varepsilon} p(x_i) y(x_i) + \frac{h}{2} q(x_{i+1}) y(x_{i+1}) + \frac{h}{2} q(x_i) y(x_i) + \frac{h}{2} [r(x_i) + r(x_{i+1})] \]

(2.15) can be rewritten in a three term recurrence relationship as follows:

\[ E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i \quad ; \quad i = l + 1, l + 2, \ldots , N - 1 \]

where

\[ E_i = -\frac{\sqrt{\varepsilon}}{2} p_i \]
\[ F_i = 1 + \frac{\sqrt{\varepsilon}}{2} p_{i+1} + \frac{h}{2} \left( 1 - \frac{\sqrt{\varepsilon}}{\varepsilon} \right) p_i + \frac{h}{2} q_i \]
\[ G_i = 1 - \frac{h}{2} \left( 1 - \frac{\sqrt{\varepsilon}}{\varepsilon} \right) p_{i+1} - \frac{h}{2} q_{i+1} \]
\[ H_i = \frac{h}{2} (r_i + r_{i+1}) \]

Here \( y_i = y(x_i) \), \( p_i = p(x_i) \), \( q_i = q(x_i) \) and \( r_i = r(x_i) \). Again we observe that (2.16) also gives us the tridiagonal system.

STEP-III. We now have from (2.9) in \([x_{i-1}, x_i]\) for \( i = 1, 2, \ldots , l-1 \); and (2.16) in \([x_i, x_{i+1}]\) for \( i = l+1, l+2, \ldots , N-1 \); we get a system of \((N-2)\) equations with \((N+1)\) unknowns. From the given boundary conditions (2.2) and (2.3) we get two equations. We need one more equation to solve for the unknowns \( y_0, y_1, \ldots , y_n \). For this we consider the original differential equation at \( x = x_l = 0 \). Since \( a(x) = 0 \) at \( x = x_l = 0 \), we get the following:

\[ \varepsilon y''(x_l) + b(x_l) y(x_l) = f(x_l) \]  

(2.17)

By making use of the second order central finite difference approximation for the second order derivative in (2.8) at \( x_l \)

\[ [\varepsilon] y_{l-1} - [2\varepsilon - h^2 b_l] y_l + [\varepsilon] y_{l+1} = h^2 f_l \]  

(2.18)

With this equation we now have \((N+1)\) equations to solve for the \((N+1)\) unknowns \( y_0, y_1, \ldots , y_n \). We solve this tridiagonal algebraic system by using an efficient and stable Thomas Algorithm.

### 2.2 Dual Boundary Layers

Consider the singular perturbation problem exhibiting twin layers of the underlying interval. In this case \( a(x) \) changes sign in the domain of interest. Without loss of generality, we can take \( a(0) = 0 \) and the interval to be \([-1, 1]\). The function \( a(x) \) is defined in \([-1, 1]\) as follows:

\[
a(x) = -x > 0 \quad \text{for} \quad -1 \leq x < 0, \\
a(x) = -x = 0 \quad \text{for} \quad x = 0, \\
a(x) = -x < 0 \quad \text{for} \quad 0 < x \leq 1,
\]

Divide the interval \([-1, 1]\) into \( N \) equal parts with mesh size \( h \), i.e., \( h = \frac{2}{N} \) and \( x_i = -1 + ih \) for \( i = 0, 1, \ldots , N \). Let us denote \( \frac{2}{h} = l \). Then, again divide the interval \([-1, 1]\) into two sub intervals \([x_i, x_{i+1}]\) for \( i = 1, 2, \ldots , l-1; [x_{i+1}, x_i] \) for \( i = l+1, l+2, \ldots , N-1 \).

STEP I: Now by using Taylor series expansion in the neighborhood of the point \( x \in [x_i, x_{i+1}] \), we have

\[
y(x - \sqrt{\varepsilon}) \approx y(x) - \sqrt{\varepsilon} y'(x) + \frac{\varepsilon}{2} y''(x)
\]
\[
y''(x) \approx \frac{2y(x - \sqrt{\varepsilon}) - 2y(x) + 2\sqrt{\varepsilon} y'(x)}{\varepsilon} \]

(2.19)
and consequently, equation (2.1) is replaced by the following first order differential equation with a small deviation argument:

\[ y'(x) = p(x)g(x - \sqrt{\varepsilon}) + q(x)y(x) + r(x), \text{ for } 1 + \sqrt{\varepsilon} \leq x \leq 0 \]  

(2.20)

where \( p(x), q(x) \) and \( r(x) \) are given by

\[
\begin{align*}
p(x) &= \frac{-2}{2\sqrt{\varepsilon + a(x)}} \\
q(x) &= \frac{2}{2\sqrt{\varepsilon + a(x)}} \\
r(x) &= \frac{f(x)}{2\sqrt{\varepsilon + a(x)}}
\end{align*}
\]

Integrating equation (2.20) in the subinterval \([x_i, x_{i+1}], i=1, 2, \ldots, \, l-1\), we get

\[
y(x_{i+1}) - y(x_i) = \int_{x_i}^{x_{i+1}} (p(x)g(x - \sqrt{\varepsilon}) + q(x)y(x) + r(x)) \, dx
\]

By making use of trapezoidal formula for evaluating the integrals approximately, we obtain

\[
y(x_{i+1}) - y(x_i) = \frac{h}{4} \left[ p(x_{i+1})y(x_{i+1}) - \sqrt{\varepsilon} + p(x_i)y(x_i) - \sqrt{\varepsilon} + q(x_{i+1})y(x_{i+1}) + q(x_i)y(x_i) + r(x_{i+1}) + r(x_i) \right]
\]

Again by means of Taylor series expansion, we have

\[
y'(x) \approx y(x) - \sqrt{\varepsilon}y'(x)
\]

And then by approximating \( y'(x) \) by linear interpolation, we get

\[
y(x_i - \sqrt{\varepsilon}) \approx y(x_i) - \sqrt{\varepsilon} \left( \frac{y(x_i) - y(x_{i-1})}{h} \right) = \left( 1 - \frac{\sqrt{\varepsilon}}{h} \right) y(x_i) + \frac{\sqrt{\varepsilon}}{h} y(x_{i-1})
\]

(2.22)

And similarly

\[
y(x_{i+1} - \sqrt{\varepsilon}) \approx y(x_{i+1}) - \sqrt{\varepsilon} \left( \frac{y(x_{i+1}) - y(x_i)}{h} \right) = \left( 1 - \frac{\sqrt{\varepsilon}}{h} \right) y(x_{i+1}) + \frac{\sqrt{\varepsilon}}{h} y(x_i)
\]

(2.23)

Substituting equations (2.21) and (2.22) in (2.23) and rearranging, we get

\[
y(x_{i+1}) - y(x_i) = \frac{h}{2} (1 - \frac{\sqrt{\varepsilon}}{h}) p(x_{i+1}) y(x_{i+1}) + \frac{\sqrt{\varepsilon}}{2} p(x_{i+1}) y(x_i) + \frac{h}{2} (1 - \frac{\sqrt{\varepsilon}}{h}) p(x_i) y(x_i) + \frac{\sqrt{\varepsilon}}{2} p(x_i) y(x_{i-1}) + \frac{h}{2} q(x_{i+1}) y(x_{i+1}) + \frac{h}{2} q(x_i) y(x_i) + \frac{h}{2} [r(x_i) + r(x_{i+1})]
\]

(2.24)

(2.24) can be rewritten in a three term recurrence relationship as follows:

\[
E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i \quad ; \quad i = 1, 2, \ldots, l - 1
\]

(2.25)

where

\[
\begin{align*}
E_i &= -\frac{\sqrt{\varepsilon}}{2} p_i \\
F_i &= 1 + \frac{\sqrt{\varepsilon}}{2} p_{i+1} + \frac{h}{2} \left( 1 - \frac{\sqrt{\varepsilon}}{h} \right) p_i + \frac{h}{2} q_i \\
G_i &= 1 - \frac{h}{2} \left( 1 - \frac{\sqrt{\varepsilon}}{h} \right) p_{i+1} - \frac{h}{2} q_{i+1} \\
H_i &= \frac{h}{2} (r_i + r_{i+1})
\end{align*}
\]

Here \( y_i = y(x_i) \), \( p_i = p(x_i) \), \( q_i = q(x_i) \) and \( r_i = r(x_i) \)
STEP II: By using Taylor series expansion in the neighborhood of the point \( x \in [x_{i-1}, x_i] \), we have
\[
y \left( x + \sqrt{\varepsilon} \right) \approx y(x) + \sqrt{\varepsilon} y'(x) + \frac{\varepsilon}{2} y''(x)
\]
and consequently, equation (2.1) is replaced by the following first order differential equation with a small deviation argument:
\[
y'(x) = p(x)y(x + \sqrt{\varepsilon}) + q(x)y(x) + r(x), \text{ for } 0 \leq x \leq 1 - \sqrt{\varepsilon}
\]
where \( p(x), q(x) \) and \( r(x) \) are given by
\[
\begin{align*}
p(x) &= \frac{-2}{-2\sqrt{x+a} + 2b(x)} \\
q(x) &= \frac{-2}{-2\sqrt{x+a} + 2b(x)} \\
r(x) &= \frac{-2}{-2\sqrt{x+a} + 2b(x)}
\end{align*}
\]
Integrating (2.27) in the subinterval \([x_{i-1}, x_i], i = 1, 2, \ldots, l-1\), we get
\[
y(x_i) - y(x_{i-1}) = \int_{x_{i-1}}^{x_i} \left( p(x)y(x + \sqrt{\varepsilon}) + q(x)y(x) + r(x) \right) \, dx
\]
By making use of trapezoidal formula for evaluating the integrals approximately, we obtain
\[
y(x_i) - y(x_{i-1}) = \frac{h}{2} \left[ p(x_{i-1}) y(x_{i-1} + \sqrt{\varepsilon}) + p(x_i) y(x_i + \sqrt{\varepsilon}) + q(x_{i-1}) y(x_{i-1}) + q(x_i) y(x_i) \right]
\]
Again by means of Taylor series expansion, we have \( y(x + \sqrt{\varepsilon}) \approx y(x) + \sqrt{\varepsilon} y'(x) \) and then by approximating \( y'(x) \) by linear interpolation, we get
\[
y(x_i + \sqrt{\varepsilon}) \approx y(x_i) + \sqrt{\varepsilon} \left( \frac{y(x_{i+1}) - y(x_i)}{h} \right)
\]
And similarly
\[
y(x_{i-1} + \sqrt{\varepsilon}) \approx y(x_{i-1}) + \sqrt{\varepsilon} \left( \frac{y(x_i) - y(x_{i-1})}{h} \right)
\]
Substituting (2.29) and (2.30) in (2.28) and rearranging, we get a three term recurrence relationship as follows:
\[
E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i \quad ; i = l + 1, l + 2, \ldots, N - 1
\]
where
\[
\begin{align*}
E_i &= -1 - \frac{b}{2} p_{i-1} - \frac{\sqrt{\varepsilon}}{2} p_{i+1} - \frac{b}{2} q_{i-1} \\
F_i &= -1 + \frac{\sqrt{\varepsilon}}{2} p_{i-1} - \frac{\sqrt{\varepsilon}}{2} p_{i+1} + \frac{b}{2} p_i + \frac{b}{2} q_i \\
G_i &= -\frac{\sqrt{\varepsilon}}{2} p_i \\
H_i &= \frac{h}{2} (r_i + r_{i-1})
\end{align*}
\]
Here \( y_i = y(x_i), \ p_i = p(x_i), \ q_i = q(x_i) \) and \( r_i = r(x_i) \) (2.31) gives us the tridiagonal system.
STEP III: We have from (2.25) in \([x_{i-1}, x_i]\) for \( i = 1, 2, \ldots, l-1 \); and (2.31) in \([x_i, x_{i+1}]\) for \( i = l+1, l+2, \ldots, N-1 \); a system of \( (N-2) \) equations with \( (N+1) \) unknowns. From the given boundary conditions (2.2) and (2.3) we get two equations. We need one more equation to solve for the
unknowns $y_0, y_1, \ldots, y_n$. For this we consider the original differential equation at $x = x_l = 0$. Since $a(x) = 0$ at $x = x_l = 0$, we get the following:

$$\varepsilon y''(x_l) + b(x_l)y(x_l) = f(x_l)$$  \hspace{1cm} (2.32)

By making use of the second order central finite difference approximation for the second order derivative in (2.32) at $x_l$

$$[\varepsilon] y_{l-1} - [2\varepsilon - h^2 b_l] y_l + [\varepsilon] y_{l+1} = h^2 f_l$$  \hspace{1cm} (2.33)

With this equation we now have $(N+1)$ equations to solve for the $(N+1)$ unknowns $y_0, y_1, \ldots, y_n$. We solve this tridiagonal algebraic system by using an efficient and stable Thomas Algorithm.

3 Numerical Examples

To demonstrate the applicability of non-iterative numerical integration method computationally, first we consider one left-end and one right-end boundary layer problem on the interval $[0, 1]$, and then we apply the method on internal layer, dual boundary layer problems. These problems have been chosen because they have been widely discussed in the literature and because exact solutions are available for comparison. We solve the problems with step lengths $h = 2^{-m}$, $m = 3, 4, 5, 6$ and 10 by our method, for different values of $\varepsilon$. The relative errors with $L_2$-norm

$$\frac{\sum (y(x_i) - y_i)^2}{\sum (y(x_i))^2}$$

are tabulated in Tables 1–7.

**Example 3.1.** Consider the following variable coefficient singular perturbation problem from Kevorkian and Cole

$$\varepsilon y''(x) + (1 - x^2)y'(x) - \frac{1}{2}y(x) = 0; \ x \in [0, 1]$$

with $y(0) = 0$ and $y(1) = 1$. Clearly, this problem has a boundary layer at $x = 0$, i.e., at the left end of the underlying interval. We have chosen to use uniformly valid approximation (which is obtained by the method given by Nayfeh, page: 148) as our ‘exact’ solution

$$y(x) = \frac{1}{(2 - x)} - \frac{1}{2} e^{-(x - x^2/4)/\varepsilon}$$

The relative errors are given in table 1 for $\varepsilon = 2^{-3}, 2^{-4}, 2^{-5}, 2^{-10}, 2^{-30}$ respectively.

**Example 3.2.** Consider the following singular perturbation problem

$$\varepsilon y''(x) - y'(x) = 0; \ x \in [0, 1]$$

with $y(0) = 0$ and $y(1) = 0$. Clearly, this problem has a boundary layer at $x = 1$, i.e., at the right end of the underlying interval. The exact solution is given by

$$y(x) = \frac{(e^{(x-1)/\varepsilon} - 1)/(e^{-1/\varepsilon} - 1)}$$

The relative errors are given in table 2 for $\varepsilon = 2^{-3}, 2^{-4}, 2^{-5}, 2^{-10}, 2^{-30}$ respectively.

**Example 3.3.** Consider the singular perturbation problem

$$\varepsilon y''(x) + 2xy'(x) = 0$$

with $y(-1) = -1$ and $y(1) = 1$. This problem has an internal layer of width $O(\sqrt{\varepsilon})$ at $x = 0$. The exact solution is given by

$$y(x) = erf\left(\frac{x}{\sqrt{\varepsilon}}\right)$$
The relative errors are given in table 3 for $\varepsilon = 2^{-3}, 2^{-4}, 2^{-5}, 2^{-10}, 2^{-30}$ respectively.

**Example 3.4.** Consider the following singular perturbation problem

$$\varepsilon y'' + xy' - y = 0, -1 \leq x \leq 1$$

with $y(-1) = 1$ and $y(1) = 2$. This problem has an internal layer of width $O(\sqrt{\varepsilon})$ at $x = 0$.

The numerical solution is given in table 4 for $\varepsilon = 2^{-3}, 2^{-5}, 2^{-10}$ respectively.

**Example 3.5.** Consider the singular perturbation problem

$$\varepsilon y''(x) + 2xy'(x) + (1 + x^2)y = 0$$

with $y(-1) = 2$ and $y(1) = 1$. This problem exhibits dual layers, that is at $x = -1$ and $x = 1$. The uniform solution is given by (Bender and Orszag, pp. 460)

$$y(x) = 2e^{-2(x+1)/\varepsilon} + e^{-2(1-x)/\varepsilon}$$

The relative errors are given in table 5 for $\varepsilon = 2^{-3}, 2^{-4}, 2^{-5}, 2^{-10}, 2^{-30}$ respectively.

**Example 3.6.** Consider the singular perturbation problem

$$\varepsilon y''(x) - 2(2x-1)y'(x) - 4y(x) = 0, \quad x \in (0, 1)$$

with $y(0)=1, y(1)=1$. This problem exhibits dual layers, that is at $x = 0$ and $x = 1$.

The exact solution is given by

$$y(x) = e^{-2x(1-x)/\varepsilon}.$$

The relative errors are given in table 6 for $\varepsilon = 2^{-3}, 2^{-4}, 2^{-5}, 2^{-10}, 2^{-30}$ respectively.

**Example 3.7.** Consider the singular perturbation problem

$$\varepsilon y'' - xy' - y = 0, \quad -1 \leq x \leq 1$$

with $y(-1) = 1$ and $y(1) = 2$. For this problem we have two boundary layers one at $x = -1$ and another at $x = 1$. The numerical solution is given in table 7 for $\varepsilon = 2^{-3}, 2^{-5}, 2^{-10}$ respectively.

### 4 Discussions and Conclusions

We have described and demonstrated the applicability of the non-iterative numerical integration method for general singularly perturbed two-point boundary value problems. This method provides an alternative and supplementary technique to the conventional ways of solving singular perturbation problems. This method is non-iterative on deviating argument, does not depend on asymptotic expansions and very easy implement on a computer to solve singular perturbation problems with a modest amount of problems preparation. We have implemented this method on examples having left, right, internal and dual boundary layers and tabulated the relative errors with $L_2$-norm obtained by present method. It can be observed that the accuracy predicted can always be achieved with very little computational effort.
### Table 1: The Relative errors in solution of Example 3.1

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<tr>
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### References


A Numerical integration method for solving singular perturbation problems

Table 5: The Relative errors in solution of Example 3.5

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Table 6: The Relative errors in solution of Example 3.6

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Table 7: Numerical result of Example 3.7

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