On the extended \((G'/G)-\)expansion method and its applications

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Abstract:

In this paper, the extended \((G'/G)-\)expansion method with a computerized symbolic computation Maple is employed for constructing the new exact travelling wave solutions of nonlinear evolution equations arising in physics. The obtained travelling wave solutions are expressed by the hyperbolic functions, the trigonometric functions and rational functions. The applied method will be used in further works to establish more entirely new solutions for other kinds of two nonlinear evolution equations, namely, the \((3+1)\)-dimensional Jimbo-Miwa equation and extended generalization of Vakhnenko equation.

Key words: Extended \((G'/G)-\)Expansion method, Nonlinear evolution equations; Travelling wave solutions.

1 Introduction

The nonlinear evolution equations in mathematical physics play a major role in various fields, such as fluid mechanics, plasma physics, optical fibers, solid state physics, chemical kinematics, chemical physics and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations.

The investigation of exact solutions of nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena and gradually becomes one of the most important and significant tasks. In the past several decades, many effective methods for obtaining exact solutions of NLEEs have been presented [1 – 26].

Our main goal in this study is to present the extended \((G'/G)-\)expansion method [10 – 12] for constructing the travelling wave solutions. In section 2, we describe the extended \((G'/G)-\)expansion method. In section 3, in order to illustrate the method we apply the method to two physically important nonlinear evolution equations, namely, the \((3+1)\)-dimensional Jimbo-Miwa equation and extended generalization of Vakhnenko equation and abundant exact solutions are obtained which included the hyperbolic functions, the trigonometric functions and rational functions. Finally, we give some concluding remarks.
2 Methodology

For a given a nonlinear equation, say in two independent variables \(x\) and \(t\) as

\[
\phi(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0, \tag{2.1}
\]

where \(u = u(x, t)\) is an unknown function, \(\phi\) is a polynomial in \(u = u(x, t)\) and its various partial derivatives, in which the highest order derivatives and nonlinear term are involved. Combining the independent variables \(x\) and \(t\) into one variable \(\xi = x - ct\), we suppose that

\[
u(x, t) = u(\xi), \xi = x - ct, \tag{2.2}\]

Eq. (1), becomes

\[
\phi(u, -cu', u', c^2u''', -cu'', \ldots) = 0 \tag{2.3}
\]

The solution of Eq. (3) can be expressed by a polynomial in \(\left(\frac{G'(\xi)}{G(\xi)}\right)\)

\[
u(\xi) = \sum_{i=-N}^{N} a_i \left(\frac{G'(\xi)}{G(\xi)}\right)^i, \tag{2.4}\]

where \(G = G(\xi)\) satisfies

\[
G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{2.5}\]

where \(G' = \frac{dG(\xi)}{d\xi}, G'' = \frac{d^2G(\xi)}{d\xi^2}\), \(a_i, \lambda\) and \(\mu\) are constants to be determined later. \(a_i \neq 0\), the unwritten part in (4) is also a polynomial in \(\left(\frac{G'}{G}\right)\), but the degree of which is generally equal to or less than \(m - 1\), the positive integer \(m\) can be determined by balancing the highest order derivative terms with nonlinear term appearing in Eq. (3).

It is worth noting that the solutions of Eq. (5) for \(\left(\frac{G'}{G}\right)\) can be written in the form of hyperbolic, trigonometric and rational functions as given below.

**The first type:** when \(\lambda^2 - 4\mu > 0\)

\[
\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{\sqrt{\lambda^2 - 4\mu}}{2} \frac{C_2 \sinh \frac{\lambda}{2} \sqrt{\lambda^2 - 4\mu} \xi + C_2 \cosh \frac{\lambda}{2} \sqrt{\lambda^2 - 4\mu} \xi - \lambda}{C_1 \cosh \frac{\lambda}{2} \sqrt{\lambda^2 - 4\mu} \xi + C_2 \sinh \frac{\lambda}{2} \sqrt{\lambda^2 - 4\mu} \xi - \lambda}; \tag{2.6}\]

**The second type:** when \(\lambda^2 - 4\mu < 0\)

\[
\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{\sqrt{4\mu - \lambda^2}}{2} \frac{-C_1 \sin \frac{\lambda}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \cos \frac{\lambda}{2} \sqrt{4\mu - \lambda^2} \xi - \frac{\lambda}{2}}{C_1 \cos \frac{\lambda}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \sin \frac{\lambda}{2} \sqrt{4\mu - \lambda^2} \xi - \frac{\lambda}{2}}; \tag{2.7}\]

**The third type:** when \(\lambda^2 - 4\mu = 0\)

\[
\left(\frac{G'(\xi)}{G(\xi)}\right) = \frac{C_2}{C_1 + C_2 \xi} - \frac{\lambda}{2}; \tag{2.8}\]

where \(C_1\) and \(C_2\) are integration constants.

Inserting Eq. (4) into (3) and using Eq. (5), collecting all terms with the same order \(\left(\frac{G'}{G}\right)\) together, the left hand side of Eq. (3) is converted into another polynomial in \(\left(\frac{G'}{G}\right)\). Equating each coefficients of this polynomial to zero, yields a set of algebraic equations for \(a_i, \lambda\) and \(\mu\). With the knowledge of the coefficients \(a_i\) and general solution of Eq. (5) we have more travelling wave solutions of the nonlinear evolution Eq. (1).
3 New Applications

To illustrate the effectiveness and convenience of the method, we consider two models arising in mathematical physics, namely, the extended generalization of Vakhnenko equation and (3+1)-dimensional Jimbo-Miwa equation.

3.1 Example[1]. The extended generalization of Vakhnenko equation

Let us first consider the extended generalization of Vakhnenko equation which can be written [27] as

\[
\frac{\partial}{\partial x} [D^2 u + \frac{1}{2} pu^2 + \beta u] + pDu = 0,
\]

where \(p\) and \(\beta\) are arbitrary non-zero constants. Eq.(9) can be traced to Vakhnenko equation (VE) which was initially presented to model high-frequency waves in a relaxing medium. When \(p = 1, \beta = 0\), Eq.(9) is reduced to Vakhnenko equation. When \(p = 1, \beta = 0\) is arbitrary non-zero constant, Eq.(9) is reduced to a generalized of VE (GVE). We make the variable transformation

\[
x = T + \int_{-\infty}^{X} U(X', T) dX',
\]

where \(u(x, t) = U(X, T)\) and \(x_0\) is a constant. \(X\) and \(T\) are two independent variables. Then Eq.(9) reduces to

\[
U_{XXT} + pU_{XT} - pU_X \int_{-\infty}^{X} U(X', T) dX' + \beta U_T + pU_X = 0
\]

(3.4)

We introduce a new function \(W\) defined by \(W(X, T) = \int_{-\infty}^{X} U(X', T) dX'\), the

\[
W_X = U
\]

(3.5)

Thus Eq.(12) becomes

\[
W_{XXXT} + p(W_{XWXT} + W_{XXW_T}) + \beta W_{XT} + pW_{XX} = 0
\]

(3.6)

To study the travelling wave solutions of Eq.(14), we take a plane wave transformation in the form

\[
W = W(\xi), \xi = x + ct, x = X, t = T,
\]

where \(c\) is a constant. Substituting Eq.(15) into (14), we have

\[
cW'''(\xi) + 2pcW'^2(\xi) + (\beta c + p)W' = 0,
\]

(3.8)

where the prime denotes the differential with respect to \(\xi\).

In view of the technique of solution, we introduce the ansatz

\[
W(\xi) = \sum_{i=-N}^{N} a_i \left( \frac{G'(\xi)}{G(\xi)} \right)^i
\]

(3.9)

where \(a_i\) are constants to be determined later. Our main goal is to solve Eq.(16) by means illustrated above. Considering the homogeneous balance between \(u'''(\xi)\) and \(u'^2(\xi)\) in Eq.(16), yields \(N = 1\), we suppose that the solution of Eq.(16) can be expressed by
\[ W(\xi) = a_0 + a_1 \left( \frac{G'(\xi)}{G(\xi)} \right) + a_{-1} \left( \frac{G'(\xi)}{G(\xi)} \right)^{-1}, \]  

(3.10)

where \( a_0, a_1, a_{-1} \) are constants to be determined later. Substituting Eq.(18) with Eq.(5) into Eq.(16), collecting the coefficients of \( \left( \frac{G'(\xi)}{G(\xi)} \right) \) we obtain a set of algebraic equations for \( a_0, a_1, a_{-1}, p \) and \( \beta \), and solving this system with the aid of Maple Package we obtain the two sets of solutions as

**case(1)**

\[ a_0 = a_0, a_{-1} = a_1, a_1 = 0, \beta = -\frac{3\mu - 4ca_{-1} + ca_{-1}\lambda^2}{ca_{-1}}, p = -\frac{3\mu}{a_{-1}} \]  

(3.11)

**case(2)**

\[ a_0 = a_0, a_{-1} = 0, a_1 = a_1, \beta = -\frac{ca_1\lambda^2 + 3 - 4ca_{-1}\mu}{ca_1}, p = -\frac{3}{a_1} \]  

(3.12)

By using Eq.(19), Eq.(18) can written as

\[ W_1(\xi) = a_0 + a_{-1} \left[ \frac{G'(\xi)}{G(\xi)} \right]^{-1}, \]  

(3.13)

\[ \xi = x + ct \]  

(3.14)

With the aid of Eq.(20), Eq.(19) admits to

\[ W_2(\xi) = a_0 + a_1 \left[ \frac{G'(\xi)}{G(\xi)} \right], \]  

(3.15)

\[ \xi = x + ct \]  

(3.16)

With the knowledge of the solution of Eq.(5) and Eqs.(21-24), we have three types of travelling wave solutions of the extended generalization of Vakhnenko equation Eq.(9) as

**The first type:** when \( \lambda^2 - 4\mu > 0 \)

\[ W_{1a}(\xi) = a_0 + a_{-1} \left[ \frac{\sqrt{\lambda^2 - 4\mu} C_1 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + C_2 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi}{C_1 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + C_2 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi} - \frac{\lambda}{2} \right]^{-1}, \]  

(3.17)

\[ W_{2a}(\xi) = a_0 + a_1 \left[ \frac{\sqrt{\lambda^2 - 4\mu} C_1 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + C_2 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi}{C_1 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + C_2 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi} - \frac{\lambda}{2} \right], \]  

(3.18)

**The second type:** when \( \lambda^2 - 4\mu < 0 \)

\[ W_{1b}(\xi) = a_0 + a_{-1} \left[ \frac{\sqrt{4\mu - \lambda^2} - C_1 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi}{C_1 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi} - \frac{\lambda}{2} \right]^{-1}, \]  

(3.19)

\[ W_{2b}(\xi) = a_0 + a_1 \left[ \frac{\sqrt{4\mu - \lambda^2} - C_1 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi}{C_1 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi} - \frac{\lambda}{2} \right], \]  

(3.20)

**The third type:** when \( \lambda^2 - 4\mu = 0 \)
\begin{equation}
W_{1c}(\xi) = a_0 + a_{-1} \left[ \frac{C_2}{C_1 + C_2 \xi} - \frac{\lambda}{2} \right]^{-1},
\end{equation}

(3.21)

\begin{equation}
W_{2c}(\xi) = a_0 + a_1 \left[ \frac{C_2}{C_1 + C_2 \xi} - \frac{\lambda}{2} \right],
\end{equation}

(3.22)

where \( C_1 \) and \( C_2 \) are integration constants.

3.2. Example[2]. The (3+1)-dimensional Jimbo-Miwa equation

A second instructive model is the (3+1)-dimensional Jimbo-Miwa equation as [28]

\begin{equation}
u_{xxxxy} + 3u_x u_{xy} + 3u_y u_{xx} + 2u_y t - 3u_z = 0,
\end{equation}

(3.23)

which comes from the second member of a KP-hierarchy called Jimbo Miwa equation firstly introduced by Jimbo-Miwa [29]. To investigate the travelling propagation solution of Eq. (31), we first introduce the following transformation

\begin{equation}
u(x, y, z, t) = \nu(\xi), \xi = x + y + z - ct,
\end{equation}

(3.24)

where \( c \) is a constant. Using transformation (32), Eq. (31) in terms of the new variable \( \xi \) reads

\begin{equation}
u''''(\xi) + 3u''(\xi) - (2c + 3)u'(\xi) = 0
\end{equation}

(3.25)

Balancing the linear term of the highest order \( u''''(\xi) \) with the nonlinear term \( u''(\xi) \), yields \( N = 1 \). Then the solution are given by

\begin{equation}
u(\xi) = a_0 + a_1 \left[ \frac{G'(\xi)}{G(\xi)} \right] + a_{-1} \left[ \frac{G'(\xi)}{G(\xi)} \right]^{-1},
\end{equation}

(3.26)

where \( a_0, a_1 \) and \( a_{-1} \) are constants to be determined later. Substituting Eq. (34) with Eq. (5) into Eq. (33), collecting the coefficients of \( \left( \frac{G'(\xi)}{G(\xi)} \right) \) we obtain a set of algebraic equations for \( a_0, a_1, a_{-1} \) and \( c \), and solving this system, we have

**case(1)**

\begin{equation}
a_{-1} = 0, a_0 = a_0, a_1 = 2, c = -2\mu - \frac{3}{2} + \frac{\lambda^2}{2}.
\end{equation}

(3.27)

**case(2)**

\begin{equation}
a_{-1} = -2\mu, a_0 = a_0, a_1 = 0, c = -2\mu - \frac{3}{2} + \frac{\lambda^2}{2}.
\end{equation}

(3.28)

According to case(1), Eq. (34) yields

\begin{equation}
u_1(\xi) = a_0 + 2\left[ \frac{G'(\xi)}{G(\xi)} \right],
\end{equation}

(3.29)

\begin{equation}\xi = x + y + z - \left[ -2\mu - \frac{3}{2} + \frac{\lambda^2}{2} \right] t
\end{equation}

(3.30)

In view of case(2), Eq. (34) yields

\begin{equation}
u_2(\xi) = a_0 - 2\mu \left[ \frac{G'(\xi)}{G(\xi)} \right]^{-1},
\end{equation}

(3.31)

\begin{equation}\xi = x + y + z - \left[ -2\mu - \frac{3}{2} + \frac{\lambda^2}{2} \right] t
\end{equation}

(3.32)
Using the solution of Eq.(5) into Eqs.(34-40), we have three types of travelling wave solutions of Eq.(34) as

**The first type:** when \( \lambda^2 - 4\mu > 0 \)

\[
\begin{align*}
\notag u_{1a}(\xi) &= a_0 + 2\left[ \frac{\sqrt{\lambda^2 - 4\mu} \ \frac{1}{2} \ \left( C_1 \sinh \frac{\lambda}{2} \sqrt{\lambda^2 - 4\mu} + C_2 \cosh \frac{\lambda}{2} \sqrt{\lambda^2 - 4\mu} \right) - \frac{\lambda}{2} \right], \\
\notag u_{2a}(\xi) &= a_0 - 2\mu \left[ \frac{\sqrt{\lambda^2 - 4\mu} \ \frac{1}{2} \ \left( C_1 \sinh \frac{\lambda}{2} \sqrt{\lambda^2 - 4\mu} + C_2 \cosh \frac{\lambda}{2} \sqrt{\lambda^2 - 4\mu} \right) - \frac{\lambda}{2} \right]^{-1}
\end{align*}
\]  

**The second type:** when \( \lambda^2 - 4\mu < 0 \)

\[
\begin{align*}
\notag u_{1b}(\xi) &= a_0 + 2\left[ \frac{\sqrt{4\mu - \lambda^2} \ \frac{1}{2} \ \left( C_1 \sin \frac{\lambda}{2} \sqrt{4\mu - \lambda^2} + C_2 \cos \frac{\lambda}{2} \sqrt{4\mu - \lambda^2} \right) - \frac{\lambda}{2} \right], \\
\notag u_{2b}(\xi) &= a_0 - 2\mu \left[ \frac{\sqrt{4\mu - \lambda^2} \ \frac{1}{2} \ \left( C_1 \sin \frac{\lambda}{2} \sqrt{4\mu - \lambda^2} + C_2 \cos \frac{\lambda}{2} \sqrt{4\mu - \lambda^2} \right) - \frac{\lambda}{2} \right]^{-1}
\end{align*}
\]  

**The third type:** when \( \lambda^2 - 4\mu = 0 \)

\[
\begin{align*}
\notag u_{1c}(\xi) &= a_0 + 2\left[ \frac{C_2}{C_1 + C_2 \xi} - \frac{\lambda}{2} \right], \\
\notag u_{2c}(\xi) &= a_0 - 2\mu \left[ \frac{C_2}{C_1 + C_2 \xi} - \frac{\lambda}{2} \right]
\end{align*}
\]

where \( C_1 \) and \( C_2 \) are integration constants.

4 Conclusion

The extended \( \left( \frac{G'}{G} \right) \)-expansion method was successfully used to establish the travelling wave solutions for nonlinear evolution equations arising in mathematical physics, namely, the \((3+1)\)-dimensional Jimbo-Miwa equation and extended generalization of Vakhnenko equation.

As a results, many exact travelling wave solutions are obtained which include the hyperbolic functions, trigonometric functions and rational functions.

It is worthwhile to mention that the proposed method is reliable and effective and gives more solutions. The applied method will be used in further works to establish more entirely new exact travelling wave solutions for treating many other nonlinear partial differential equations of mathematical physics.

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