Controllability of Discrete Volterra Systems

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Abstract:
In this paper, necessary and sufficient condition is established for controllability of discrete time linear Volterra systems. Local controllability result for a semi-linear discrete Volterra system is also proved. Numerical examples are provided to illustrate our results.

Key words: Controllability, Controllability Grammian, Higher order functions, Semi-linear system

1 Introduction
In [2], Gaishun and Dymkov studied the controllability of the Volterra linear discrete system

\[ x(t + 1) = \sum_{i=0}^{t} A(i)x(t-i) + Bu(t), \quad t \in N_0 \triangleq \{0, 1, 2, \ldots\} \]

by a method based on the representation of the Volterra operator generated by the equation in the ring of formal power series. In this paper we study controllability of a non-autonomous linear system of the form:

\[ \Sigma_L : x(t + 1) = \sum_{i=0}^{t} A(i)x(t-i) + B(t)u(t), \quad t \in N_0 \tag{1.1} \]

and local controllability of a semi-linear discrete Volterra system of the form:

\[ \Sigma_N : x(t + 1) = \sum_{i=0}^{t} A(i)x(t-i) + B(t)u(t) + f(x(t), u(t)), \quad t \in N_0 \tag{1.2} \]

using a different approach and in much more straightforward manner. Here, \((A(t))_{t \in N_0}\) and \((B(t))_{t \in N_0}\) are sequences of real \(n \times n\) and \(n \times m\) - matrices, respectively, and \((x(t))_{t \in N_0}\) and \((u(t))_{t \in N_0}\) are sequences of state vectors in \(R^n\) and control vectors in \(R^m\), respectively. \(f(\ldots) : R^n \times R^m \rightarrow R^n\) is a nonlinear function of state and control variables. It follows easily that for a given control sequence \(\{u(t)\}_{t \in N_0}\) and initial state \(x(t_0) = x_0\), there exists a unique solution to the
linear system $\Sigma_L$.

We make the following definitions to obtain solution of $\Sigma_L$. Define the set of linear operators $Q_t : \mathbb{R}^n \to \mathbb{R}^n$, $t \in \mathbb{N}_0$ by

$$Q_0 = I, \quad Q_{t+1} = \sum_{i=0}^{t} A(i)Q_{t-i}, \quad t \in \mathbb{N}_0$$

(1.3)

Using these operators, the solution of (1.1) is given by

$$x(t) = Q_t x_0 + \sum_{i=0}^{t-1} Q_i B(t - i - 1)u(t - i - 1)$$

(1.4)

We first obtain the controllability results of the linear system $\Sigma_L$.

**Definition 1.1.** (Global controllability [6]) Let $x_0, x_1 \in \mathbb{R}^n$ be given arbitrarily. The system $\Sigma_L$ is controllable if we can find a sequence of control vectors $\{u(t) \in \mathbb{R}^m, \ t \in \mathbb{N}_0\}$ such that for some $N \in \mathbb{N}_0$ the solution $(x(t))_{t \in \mathbb{N}_0}$ of equation (1.1) with $x(0) = x_0$ satisfies the desired final state

$$x(N) = x_1$$

(1.5)

In view of (1.4), we are looking for a control sequence $(u(t))_{t \in \mathbb{N}_0}$ satisfying

$$x_1 - Q_N x_0 = \sum_{i=0}^{N-1} Q_i B(N - i - 1)u(N - i - 1)$$

or

$$x_1 - Q_N x_0 = \sum_{i=1}^{N} Q_{i-1} B(N - i)u(N - i)$$

(1.6)

This is a linear system for the unknowns $\{u(0), ..., u(N - 1)\} \in \mathbb{R}^{mN}$. Define

$$U \triangleq \{u = (u(0), u(1), ..., u(N - 1)) \in \mathbb{R}^{mN}\}$$

Equation (1.6) shows that there will be an input $u \in U$ that will transfer a given arbitrary state $x_0$ to a desired final state $x_1$ in $N$ time steps if and only if the linear map $L : \mathbb{R}^{mN} \to \mathbb{R}^n$ defined by

$$L : u \to \sum_{i=1}^{N} Q_{i-1} B(N - i)u(N - i)$$

(1.7)

is onto. From (1.7), we see that $LL^* : \mathbb{R}^n \to \mathbb{R}^n$ has a square matrix representation, where the adjoint operator $L^*$ is defined as follows:
For \( v \in \mathbb{R}^n, u \in U \),
\[
< Lu, v > = \sum_{i=1}^{N} Q_{i-1} B(N - i) u(N - i), v >
\]
\[
= \sum_{i=1}^{N} < Q_{i-1} B(N - i) u(N - i), v >
\]
\[
= \sum_{i=1}^{N} < B(N - i) u(N - i), Q_{i-1}^* v >
\]
\[
= \sum_{i=1}^{N} < u(N - i), B^*(N - i) Q_{i-1}^* v >
\]
\[
= < u, (B^* Q^*) v >
\]
\[
i.e. \quad < u, L^* v > = < u, (B^* Q^*) v >
\]
\[
i.e. \quad L^* v = (B^*(N - 1) Q_0^* v, B^*(N - 2) Q_1^* v, ..., B^*(0) Q_{N-1}^* v)
\]
\[
i.e. \quad LL^* v = \sum_{i=1}^{N} Q_{i-1} B(N - i) B^*(N - i) Q_{i-1}^* v
\]

Now define the controllability Grammian for the linear Volterra system (1.1) by
\[
W(0, N) = \sum_{i=1}^{N} Q_{i-1} B(N - i) B^*(N - i) Q_{i-1}^*
\]

In section 2, we give two different conditions, for the global controllability of (1.1), namely

(i) condition using Controllability Grammian and
(ii) Kalman type rank condition.

In Section 3, we prove a local controllability theorem for the semi-linear system (1.2) and numerical examples illustrating our results are included in Section 4.

2 Controllability of Linear System

In this section we prove necessary and sufficient condition for the controllability of the linear Volterra system. We also provide algorithm for the computation of the steering controls.

**Theorem 2.1.** Let \((A(t))_{t \in \mathbb{N}_0}\) and \((B(t))_{t \in \mathbb{N}_0}\) be sequences of real \(n \times n\) and \(n \times m\) - matrices, respectively and let \(L\) be the operator defined as in (1.7). Then the following statements are equivalent.

1. The non-autonomous Volterra system (1.1) is controllable on \([0, N]\)
2. \(\text{range}(L) = \mathbb{R}^n\).
3. \(\text{range}(LL^*) = \mathbb{R}^n\).
4. \(\text{det} W(0, N) \neq 0\), where \(W\) is the Controllability Grammian defined by (1.8).

**Proof:** The solution of the system (1.1) is given by
\[
x(t) = Q_tx_0 + \sum_{i=0}^{t-1} Q_i B(t - i - 1) u(t - i - 1)
\]
We now prove (1) ⇔ (2).
The system (1.1) is controllable on \([0, N]\) if and only if for every \(x_1\) and \(x_0 \in \mathbb{R}^n\) there exists a control sequence \(u \in U\) satisfying

\[
x_1 = Q_N x_0 + \sum_{i=0}^{N-1} Q_i B(N - i - 1) u(N - i - 1)
\]
or

\[
x_1 - Q_N x_0 = \sum_{i=1}^N Q_{i-1} B(N - i) u(N - i)
\]

Thus the system (1.1) is controllable if and only if the operator \(L : \mathbb{R}^{mN} \to \mathbb{R}^n\) defined by (1.7) is surjective. Thus statement (1) is equivalent, as noted above to the surjectivity of \(L\), which is equivalent to \(\text{range}(L) = \mathbb{R}^n\).

(2) ⇔ (3)
Let \(\text{range}(L) = \mathbb{R}^n\).
⇔ \(\text{range}(LL^*) = \mathbb{R}^n\).

(3) ⇔ (4)
From (1.8), it follows that \(LL^* = W\) is a square matrix and hence
\[
\text{range}(LL^*) = \mathbb{R}^n,
\]
⇔ \(W(0, N)\) is invertible.
⇔ \(\det(W(0, N)) \neq 0\).

Hence the proof.

Let us define the controllability matrix by
\[
W_c = [B(N - 1) \mid Q_1 B(N - 2) \mid \ldots \mid Q_{N-1} B(0)]
\]
and assume that
\[
\text{rank}(W_c) = \text{rank}([B(N - 1) \mid Q_1 B(N - 2) \mid \ldots \mid Q_{N-1} B(0)]) = n \tag{2.1}
\]

**Theorem 2.2.** If the rank condition (2.1) is satisfied for some \(N \in \mathbb{N}_0\), then for every pair \(x_0, x_1 \in \mathbb{R}^n\) there exists a control sequence \(\{u\}\) steering \(x_0\) to \(x_1\) in \(N\) time steps.

**Proof:** Let us assume that for some \(N \in \mathbb{N}_0\), it is true that \(\text{rank} W_c = n\). Then the system
\[
x_1 - Q_N x_0 = \sum_{i=1}^N Q_{i-1} B(N - i) u(N - i) \tag{2.2}
\]
has a solution \(u(0), \ldots, u(N - 1) \in \mathbb{R}^m\) for every choice of \(x_0, x_1 \in \mathbb{R}^n\). If the rank condition (2.1) is satisfied, the system (2.2) has infinitely many solutions. Now pick up a special one which is defined uniquely. For that purpose we define for every \(k = 1, \ldots, N\) an \(n \times m\) matrix \(C^k\) by

\[
C^k = Q_{k-1} B(N - k) \quad \text{for} \quad k = 1, 2, 3, \ldots, N
\]
The condition (2.1), then implies the existence of \(n\) linearly independent columns in matrix \(W_c\). We define a \(n \times n\) matrix \(C\) by these linearly independent columns,

\[
C = \begin{pmatrix}
C_{1j_1}^{k_1} & \cdots & C_{1j_n}^{k_n} \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots \\
C_{nj_1}^{k_1} & \cdots & C_{nj_n}^{k_n}
\end{pmatrix}
\]
Now define a control sequence \( u \in \mathbb{R}^n \) by
\[
u = \begin{pmatrix} u_{j_k, (k_1 - 1)} \\ \vdots \\ u_{j_n, (k_n - 1)} \end{pmatrix}
\]
and make
\[ u_j(k - 1) = 0 \] for \( k \neq k_1, j \neq j_k, l = 1, \ldots, n \)
then we obtain from equation (2.2)
\[ x_1 = Q_N x_0 + C u \]
Since \( C \) is invertible we have
\[ u = C^{-1} (x_1 - Q_N x_0) \] (2.3)
Obviously this control steers the system from \( x_0 \) to \( x_1 \) in \( N \) time steps. We now provide another steering control using controllability Grammian.

**Theorem 2.3.** If the system \( \Sigma_L \) is controllable, in \( N \) time steps then \( \forall x_0, x_1 \in \mathbb{R}^n \), \( \exists \) a control sequence \( u : N_0 \rightarrow \mathbb{R}^m \) defined by
\[
\{u(t - i)\}_{i=1}^{t} := \{B^*(t - i)Q^*_{i-1}W^{-1}(0, N)(x_1 - Q_N x_0)\}_{i=1}^{t}, t = 1, 2, 3, \ldots, N
\] (2.4)
steers the initial state \( x_0 \) to the desired final state \( x_1 \) in \( N \) time steps.

**Proof:** Since the linear system (1.1) is controllable, we have by Theorem 2.1, \( \det W(0, N) \neq 0 \), where \( W(0, N) \) is given by (1.8). To prove that control given by (2.4) steers the state \( x_0 \) to \( x_1 \), we substitute this control in (1.6) to obtain :
\[
x(t) = Q_t x_0 + \sum_{i=1}^{t} Q_{t-1} B(t - i) u(t - i)
\]
\[
x(t) = Q_t x_0 + \sum_{i=1}^{t} Q_{t-1} B(t - i) B^*(t - i) Q^*_{i-1} W^{-1}(0, t)(x_1 - Q_N x_0)
\]
It can be easily verified that at \( t = 0 \), \( x(0) = x_0 \) and at \( t = N \), \( x(N) = x_1 \). Hence the proof.

### 3 Controllability of Semi-linear System

Also to prove local controllability of the semi-linear system (1.2), we use the notion of higher order functions which is introduced below (refer [3]):

**Definition 3.1.** A continuously differentiable function \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is called a "higher order function", if
1. \( F(0) = 0 \)
2. \( \frac{\partial F}{\partial x}/_{x=0} = 0 \)

We denote the class of higher order functions by \( H \) and use the following properties of the higher order functions which can be verified in a straightforward manner.

1. If \( P \) is a \( n \times n \) constant matrix and \( F(.) \in H \), then \( PF(.) \in H \).
2. If $F_1, F_2 \in H$, then $F_1 + F_2 \in H$.

3. If $F_1 \in H$ and $F_2(0) = 0$ and is continuously differentiable then the composition $F_1(F_2(\cdot)) \in H$.

We now present in the following propositions specialized forms for inverse function theorem and the implicit function theorem (refer [4]) which will be used for obtaining controllability result for the semi-linear system $\Sigma_N$.

**Proposition 3.1.** (Inverse function Theorem [4]) In some neighborhood $U_1 \subset \mathbb{R}^n$ of the origin, let

$$Px + f(x) = y, \quad x \in U_1, y \in \mathbb{R}^n$$

where $P$ is a $n \times n$ nonsingular matrix and $f(\cdot) \in H$. Then there exists an open set $U_2 \subset U_1$ containing the origin such that the set $V$, defined by

$$V \equiv PU_2 + f(U_2)$$

is open, and for all $x \in U_2$ there exist $g(\cdot) \in H$ such that

$$x = P^{-1}y + g(y), \quad y \in V$$

**Proposition 3.2.** (Implicit function Theorem [4]) Let $U_1$ be an open subset of $\mathbb{R}^{n+k}$ containing the origin. Let an element of $U_1$ be denoted by $(x, y)$ with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^k$. Let $F : U_1 \rightarrow \mathbb{R}^n$ be a function defined by $F(x, y) = Px + Qy + f(x, y)$, where $P$ is nonsingular $n \times n$ matrix, $Q$ is any $k \times k$ matrix and $f(\cdot) \in H$. Then there exists an open set $U_2 \subset \mathbb{R}^k$ and $g(\cdot) \in H$ containing the origin such that

$$x = -P^{-1}Qy + g(y), \quad y \in U_2$$

satisfies the equation $F(x, y) = 0$.

By the above two propositions, when the underlying nonlinear functions are expressed as the sum of linear and higher order functions, the application of inverse function theorem and implicit function theorem becomes a matter of simple manipulation of equations. Note also that the vector $y$ can be a concatenation of several vectors. This leads to the following corollary (see [4]).

**Corollary 3.1.** If $Px + Qy + f(x, y) = z, f(\cdot) \in H$, and $P$ is nonsingular, then there exists $\tilde{g}(\cdot) \in H$ such that locally $x = P^{-1}(z - Qy) + \tilde{g}(y, z)$.

For the nonlinear system $\Sigma_N$, represented by (1.2), the following definition of local controllability is relevant.

**Definition 3.2.** (Local controllability) A system is locally controllable if there exists a neighborhood $\Omega$ of the origin such that, for any $x_0, x_1 \in \Omega$ there is a sequence of inputs $u = (u(0), u(1), ..., u(N-1))$ that steers the system from $x_0$ to $x_1$.

Now we prove the following result for the local controllability of (1.2) under the assumption that $\Sigma_L$ is controllable and the nonlinear function $f$ is of ”higher order”.

**Theorem 3.1.** If the linear system $\Sigma_L$ is controllable and $f \in H$, then the semi-linear system $\Sigma_N$ is locally controllable.

**Proof:** We will show the existence of a control sequence $u$ that transfers the state from $x_0$ to $x_1$ in $N$ time steps. The properties of higher order functions will be repeatedly used in in the
following derivations.

\[
x(1) = A(0)x(0) + B(0)u(0) + f(x(0), u(0))
\]
\[
x(2) = (A^2(0) + A(1))x(0) + A(0)B(0)u(0) + B(1)u(1)
+ A(0)f(x_0, u(0)) + f(A(0)x_0 + B(0)u(0) + f(x_0, u(0)), u(1))
\equiv Q_2x_0 + Q_1B(0)u(0) + B(1)u(1) + f_2(x_0, u(0), u(1)),
\]
taking
\[
A(0)f(x_0, u(0)) + f(A(0)x_0 + B(0)u(0)) = f_2(x_0, (u(0), u(1)))
\]
\[
x(N) = Q_Nx_0 + [B(N - 1) \mid Q_1B(N - 2) \mid \ldots \mid Q_{N-1}B(0)]
\left[\begin{array}{c}
u(N-1) \\
u(N-2)
\end{array}\right]
\]
\[
+ f_N(x_0, (u(0), u(1), \ldots, u(N-1)))
\]

Since
\[
W_c = [B(N - 1) \mid Q_1B(N - 2) \mid \ldots \mid Q_{N-1}B(0)]
\]
and
\[
u = [u(0), u(1), \ldots, u(N-1)]^T
\]
We have
\[
x(N) = Q_Nx_0 + W_cu + f_N(x_0, u)
\]
Since we require \(x(N) = x_1\),
\[
x_1 = Q_Nx_0 + W_cu + f_N(x_0, u)
\]
where \(f_2(\cdot), \ldots, f_N(\cdot)\) are all properly defined higher order functions. Let \(W^*_c\) be the transpose of \(W_c\) and let \(u = W^*_c v\). Therefore, we have
\[
x_1 = Q_Nx_0 + W_cW^*_c v + f_N(x_0, W^*_c v)
\]
Since the linear system is controllable, \(W_c\) is of full rank and hence \(W_cW^*_c\) is an invertible matrix. By Corollary 3.1, if \(x_0, x_1\) in a neighborhood \(\Omega\) of the origin, there exist \(v\) given by
\[
v = (W_cW^*_c)^{-1}(x_1 - Q_Nx_0) + g(x_0, x_1), \text{ for some } g(.) \in H.
\]
Thus the control sequence
\[
u = W^*_c v
\]
steers \(x_0\) to \(x_1\) for all \(x_0, x_1\) in a neighborhood of origin. Hence the theorem.

4 Numerical Examples

Example 1. Consider a 2-dimensional discrete linear Volterra system of the form
\[
x(t + 1) = \sum_{i=0}^{t} A(i)x(t - i) + B(t)u(t), \text{ } t \in \mathbb{N}_0
\]
with \( A(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ \frac{1}{t} & 2\cos(t) \end{pmatrix} \) and \( B(t) = \begin{pmatrix} .5 \\ .5t \end{pmatrix} \). Let us take \( N = 5 \) and initial state \( x_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) and the final state \( x_1 = \begin{pmatrix} -20 \\ 1 \end{pmatrix} \). Then using the Matlab program we compute the value of controllability matrix

\[
W_c = \begin{pmatrix}
0.5000 & 0.5000 & 1.6116 & 2.5491 & 0.6294 \\
2.0000 & 3.0000 & 5.1806 & 6.2451 & 1.7621
\end{pmatrix}
\]

This shows that rank of \( W_c \) is 2 and hence the given system is controllable by Theorem 2.2. Hence we can compute the control that steers initial state \( x_0 \) to desired final state \( x_1 \). Computation of this control is done using the formula (2.3), where matrix \( C \) is computed as

\[
C = \begin{pmatrix}
0.5000 & 0.5000 \\
2.0000 & 3.0000
\end{pmatrix}
\]

. Using this value of \( C \) and other known values, we obtain control \( u = \begin{pmatrix} -17.3487 \\ 26.4393 \\
0 \\
0 \\
0
\end{pmatrix} \). Figure 1 shows the controlled trajectory using this control.

Example 2. In this example we take all the data similar to the previous example and use the techniques of Theorems 2.1 and 2.3 to compute steering control. According to this, if \( \det W(0, N) \neq 0 \) then the system \( \Sigma_L \) is controllable and the control sequence which steers the initial state to final state is given by equation (2.4).

For the same data, we get controllability gramian matrix as \( W(0, 5) = \begin{pmatrix} 9.9913 & 27.8775 \\ 27.8775 & 81.9444 \end{pmatrix} \). Its determinant is \( \det W(0, 5) = 41.5806 \neq 0 \). Hence linear system is controllable. The control sequence computed using (2.4) is given by

\[
u = (4.8698, 12.5442, 5.9818, -5.4964, 0.3316)
\]
Using this control, we see that $x_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is steered to the final state $x_1 = \begin{pmatrix} -20 \\ 1 \end{pmatrix}$ in 5 time steps, see Figure 2.

Note that the control using the Grammian matrix is a minimum norm control.

**Example 3.** Consider the nonlinear discrete Volterra system

$$x(t + 1) = \sum_{i=0}^{t} A(i)x(t - i) + B(t)u(t) + f(x(t), u(t)) \quad t \in N_0$$

where $A(t)$ and $B(t)$ are as defined in example 1 and

$$f(x(t), u(t)) = \begin{pmatrix} x_2(t)\sin(x_1(t))u(t) \\ x_1(t)(1 - \cos(x_2(t)))u(t) \end{pmatrix}$$

Since

$$f(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial x} = \begin{pmatrix} x_2(t)\cos(x_1(t))u(t) & \sin(x_1(t))u(t) \\ (1 - \cos(x_2(t)))u(t) & x_1(t)\sin(x_2(t))u(t) \end{pmatrix},$$

obviously $(\frac{\partial f}{\partial x})_{x=0} = 0$. Thus the nonlinear function $f$ is of higher order. As discussed in Example 1, the linear system is controllable, hence by using Theorem 3.1, we find that the semi-linear system is also controllable in the neighborhood of the origin.

**References**


