Asymptotic growth of the spectral radii of collocation matrices approximating elliptic boundary value problems

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ABSTRACT
Throughout this paper we consider the Poisson equations in two dimensions. By the collocation methods based on radial basis functions and by exploiting some tools in literature: Perron-Frobenius theory and Weyl Tyrtyshnikov equal distribution, we prove under suitable assumptions on the shape parameter appearing in the radial basis functions that, the spectral radii of the collocation matrices grow as the size of the matrices, that is, \( \lim_{n \to \infty} \frac{\rho_n}{d_n} = \text{constant} \) where \( \rho_n \) and \( d_n \) are respectively the spectral radius and the size of the collocation matrix.

Keywords: radial basis functions; elliptic boundary value problems; collocation matrices; Block Toeplitz matrices; Perron-Fobenius theory; Weyl Tyrtyshnikov equal distribution; spectral radii; asymptotic growth.

1. Introduction
The purpose of this paper is to provide a deep study of the spectral radii of collocation matrices approximating elliptic boundary value problems in two dimensional case

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f \quad \text{if } (x, y) \in \Omega = (0, 1)^2 \\
u(x, y) = g(x, y) \quad \text{for } (x, y) \in \partial \Omega
\] (1.1)

The chosen method for approximating is based on the radial basis functions. These types of approximations are often very useful for obtaining a numerical solution of certain PDEs. Under certain conditions, the convergence is very fast (exponential in the number of grid points) when compared with Finite Difference or Finite Elements. The price that is paid is often an extreme ill-conditioning of the resulting structured matrices. A main role for approximation space is played a radial function and this space is made by translating a standard radial function with zero as its center. Some of the most commonly used radial basis functions are:

- Direct Multiquadric (MQ): \( \phi(t) = (t^2 + c^2)^\frac{1}{2} \)
- Inverse Multiquadric (IMQ): \( \phi(t) = (t^2 + c^2)^{-\frac{1}{2}} \)
- Gaussian: \( \phi(t) = e^{-\frac{t^2}{c^2}} \)

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where the parameter \( c \) is often called \textbf{shape parameter} and whose the role is to determine the accuracy and the stability. Denoting by \( \Omega \) an opened domain of model problem (1.1), \( \partial \Omega \) its boundary and \( \tilde{\Omega} = \partial \Omega \cup \Omega \) an artificial domain greater than real domain \( \Omega \), \( h = \frac{1}{n+1} \) the maximal step size and, 
\[
\{(x_i, y_j) = (ih, jh)\}_{i,j=0}^{n+1}
\]
the selected points that are chosen out of the real domain \( \Omega \) and are in the artificial domain, we present an interesting method using the nodes that most of them are selected out of the real domain and the others in the domain. Hence, the real collocation matrix is the sum of a symmetric block Toeplitz matrices with symmetric Toeplitz blocks of size \( d_n = (n + 2)^2 \) generated by unbounded function over \( \Omega \) and a matrix of rank at most \( 4n + 4 \), which collects the boundary conditions. 

More in detail we are interested in fast solution methods, and especially in the asymptotic growth of the spectral radii of the resulting matrices and in the global distribution results. For the latter point, we need to think not to a single linear system but to a sequence of linear systems of increasing dimensions \( d_n^2 \) related to a finesse parameter, as usually occurs in the approximation of PDEs. Such kind of spectral knowledge is then employed for suggesting appropriate \( O(d_n^2 \log d_n) \) preconditioners for krylov subspace methods. A first important step for understanding the spectral behavior of the considered matrices done in [1], where the link with Toeplitz sequences generated by a symbol was exploited. One of the advantages of meshless methods based on radial functions with respect to others, is high decreased of computational volume that arises when changing multi- dimensions to one dimension. Kansa [4] is the first researcher that applied an approximation by radial basis functions (Pseudo interpolation) to PDEs. The use of the globally supported radial functions, reaches to the large linear systems, poorly condition number and full matrices.

The paper is organized as follows. In section 2, we recall some useful results due to the Perron-Frobenius theory and to the Weyl Tyrtyshnikov equal distribution. Section 3 deals with the approximation of Poisson equation (1.1) by the collocation systems. The approximation of collocation sequences by the sequences of block Toeplitz matrices and the asymptotic behavior of the generating function of Toeplitz sequences are established in section 4. Section 5 is reserved to the asymptotic growth of the spectral radii of collocation matrices while section 6 deals with the general conclusions and future works.

2. Some Definitions and main results

The aim of this section is to recall some definitions and main results of the linear algebra which are useful for the study of the collocation matrices.

2.1 Perron Frobenius theory

Throughout this subsection, we recall the Perron Frobenius theory.

\textbf{Definition 2.1.} Let \( A = (a_{jk}) \) and \( B = (b_{jk}) \) be two \( n \times r \) matrices. Then, \( A \geq B \) \( (A > B) \) if \( a_{jk} \geq b_{jk} \) \( (a_{jk} > b_{jk}) \) for all \( j = 1, 2, \ldots, n \) and \( k = 1, 2, \ldots, r \).

\textbf{Definition 2.2.} \( A \in \mathbb{R}^{n \times r} \) is said to be nonnegative (positive) matrix if \( A \geq 0 \) \( (A > 0) \).

\textbf{Definition 2.3.} A matrix \( A \in \mathbb{R}^{n \times n} \) possesses the Perron-Frobenius property if its dominant eigenvalue \( \lambda_1 \) is positive and the corresponding eigenvector \( x^{(1)} \) is nonnegative.

\textbf{Definition 2.4.} A matrix \( A \in \mathbb{R}^{n \times n} \) possesses the strong Perron-Frobenius property if its dominant eigenvalue \( \lambda_1 \) is positive, simple \( (\lambda_1 > |\lambda_j|, j = 2, 3, \ldots, n) \) and the corresponding eigenvector \( x^{(1)} \) is positive.

\textbf{Definition 2.5.} A matrix \( A \in \mathbb{R}^{n \times n} \) is said to be eventually positive (eventually nonnegative) if there exists a positive integer \( k_0 \) such that \( A^k > 0 \) \( (A^k \geq 0) \) for all \( k \geq k_0 \).

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Theorem 2.1. [5, 7, 13]. For a symmetric matrix $A \in \mathbb{R}^{n \times n}$, the following properties are equivalent:

(i) $A$ possesses the strong Perron-Frobenius property.

(ii) $A$ is an eventually positive matrix.

Proof. ($i \Rightarrow ii$) : $\lambda_1 = \rho(A) > |\lambda_2| \geq \ldots \geq |\lambda_n|$, where $\lambda_1$ is a simple eigenvalue with the eigenvector $x^{(1)} \in \mathbb{R}^n$ being positive. Choose the $i$-th column $a^{(i)} \in \mathbb{R}^n$ of $A$.

Expand $a^{(i)}$: $a^{(i)} = \sum_{j=1}^{n} c_j x^{(j)}$ (where $\{x^{(1)}, x^{(2)}, \ldots, x^{(n)}\}$ is an orthonormal basis of $\mathbb{R}^n$).

$c_j = (a^{(i)}, x^{(j)})$, $j = 1, 2, \ldots, n$. So, $c_1 = (a^{(i)}, x^{(1)}) = \lambda_1 x_1^{(1)} > 0$. Apply power method: $\lim_{k \to \infty} A^k a^{(i)} > 0 \Rightarrow A^k a^{(i)} > 0 \ \forall k > m$. Choose $m_0 = \min \{m : A^k a^{(i)} > 0 \ \forall k \geq m\}$, then, $A^k > 0 \ \forall k \geq k_0 = m_0 + 1$. So, $A$ is an eventually positive matrix.

$\left( ii \Rightarrow i \right)$: From the Perron-Frobenius theory of nonnegative matrices, the assumption $A^k > 0$ means that the dominant eigenvalue of $A^k$ is positive and the only one in the circle while the corresponding eigenvector is positive. It is well known that the matrix $A$ has as eigenvalues the $k$-th roots of those of $A^k$ with the same eigenvectors. Since this happens $\forall k \geq k_0$, $A$ possesses the strong Perron-Frobenius property.

Theorem 2.2. [5, 7, 13]. For a matrix $A \in \mathbb{R}^{n \times n}$ the following properties are equivalent:

i. Both matrices $A$ and $A^T$ possess the strong Perron-Frobenius property.

ii. $A$ is an eventually positive matrix.

iii. $A^T$ is an eventually positive matrix.

Proof. ($i \Rightarrow ii$) : Let $A = XD X^{-1}$ be the Jordan canonical form of the matrix $A$. We assume that the eigenvalue $\lambda_1 = \rho(A)$ is the first diagonal entry of $D$. So the Jacobi canonical form can be written as

$$A = [x^{(1)}|X_{n,n-1}] \begin{bmatrix} \lambda_1 & 0 \\ 0 & D_{n-1,n-1} \end{bmatrix} \begin{bmatrix} y^{(1)^T} \\ Y_{n-1,n} \end{bmatrix} \tag{2.1}$$

where $y^{(1)^T}$ and $Y_{n-1,n}$ are the first row and the matrix formed by the last $n-1$ rows of $X^{-1}$, respectively. Since $A$ possesses the strong Perron-Frobenius property, the eigenvector $x^{(1)}$ is positive. From (2.1), the block form of $A^T$ is

$$A^T = [y^{(1)}|Y_{n,n-1}^T] \begin{bmatrix} \lambda_1 & 0 \\ 0 & D_{n-1,n-1}^T \end{bmatrix} \begin{bmatrix} x^{(1)^T} \\ X_{n-1,n}^T \end{bmatrix} \tag{2.2}$$

The matrix $D_{n-1,n-1}^T$ is the block diagonal matrix formed by the transpose of the Jordan blocks except $\lambda_1$. It is obvious that there exists a permutation matrix $P \in \mathbb{R}^{(n-1) \times (n-1)}$ such that the associated permutation transformation on the matrix $D_{n-1,n-1}^T$ transposes all the Jordan blocks.

Thus, $D_{n-1,n-1} = P D_{n-1,n-1}^T P$ and relation (2.2) takes the form:

$$A^T = [y^{(1)}|Y_{n,n-1}^T] \begin{bmatrix} 0 & 0 \\ P & P^T \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & D_{n-1,n-1}^T \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} x^{(1)^T} \\ X_{n-1,n}^T \end{bmatrix}$$

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where $Y_{n-1,n}^T = Y_{n-1}^T P$ and $X_{n,n-1}^T = P^T X_{n-1,n}^T$. The last relation is the Jordan canonical form of $A^T$ which means that $y^{(1)}$ is the eigenvector corresponding to the dominant eigenvalue $\lambda_1$. Since $A^T$ possesses the strong Perron-Frobenius property, $y^{(1)}$ is a positive vector or a negative one. Since $y^{(1)}^T$ is the first row of $X^{-1}$, we have that $(y^{(1)}, x^{(1)}) = 1$ implying that $y^{(1)}$ is a positive vector. We return now to the Jordan canonical form (2.1) of $A$ and form the power $A^k$.

$$A^k = [x^{(1)} | X_{n,n-1}] \begin{bmatrix} \lambda_1^k & 0 \\ 0 & D_{n-1,n-1}^k \end{bmatrix} \begin{bmatrix} y^{(1)}^T \\ Y_{n-1,n}^T \end{bmatrix}$$

then

$$\frac{1}{\lambda_1^k} A^k = [x^{(1)} | X_{n,n-1}] \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\lambda_1^k} D_{n-1,n-1}^k \end{bmatrix} \begin{bmatrix} y^{(1)}^T \\ Y_{n-1,n}^T \end{bmatrix}$$

Since $\lambda_1$ is the dominant eigenvalue, the only one of modulus $\lambda_1$, we get that $\lim_{k \to \infty} \frac{1}{\lambda_1^k} D_{n-1,n-1}^k = 0$. Thus

$$\lim_{k \to \infty} \frac{1}{\lambda_1^k} A^k = x^{(1)} y^{(1)}^T > 0.$$
\[ y^T A e = \sum_{i=1}^{n} \left( y_i \sum_{j=1}^{n} a_{ij} \right) \geq \min_{1 \leq i \leq n} \left( \sum_{j=1}^{n} a_{ij} \right) \sum_{i=1}^{n} y_i. \]

On the other hand, we get
\[ y^T A e = e^T A e = \rho(A) y^T = \rho(A) \sum_{j=1}^{n} y_j. \]

Combining the relations above, we get our result. Obviously, equality holds if the row sums are equal. If \( A^T \) possesses the strong Perron-Frobenius property, then \( y > 0 \) and the inequalities become strict.

**Corollary 2.5.** [5, 7, 13]. If \( A \in \mathbb{R}^{n \times n} \) possesses the Perron-Frobenius property, then either
\[ \sum_{i=1}^{n} a_{ij} = \rho(A) \quad \forall j = 1, 2, ..., n, \] (2.5)
or
\[ \min_{1 \leq j \leq n} \left( \sum_{i=1}^{n} a_{ij} \right) \leq \rho(A) \leq \max_{1 \leq j \leq n} \left( \sum_{i=1}^{n} a_{ij} \right). \] (2.6)

Moreover, if \( A \) possesses the strong Perron-Frobenius property, then both inequalities in (2.6) are strict.

**Theorem 2.6.** [5, 7, 13]. If the matrices \( A, B \in \mathbb{R}^{n \times n} \) are such that \( A \leq B \), and both \( A \) and \( B^T \) possess the Perron-Frobenius property (or both \( A^T \) and \( B \) possess the Perron-Frobenius property), then
\[ \rho(A) \leq \rho(B). \] (2.7)

Moreover, if the above matrices possess the strong Perron-Frobenius property and \( A \neq B \) then, the inequality (2.7) becomes strict.

**Proof.** Let \( x, y \geq 0 \) be the Perron right and left eigenvectors of \( A \) and \( B \) associated with the dominant eigenvalues \( \lambda_A \) and \( \lambda_B \), respectively. Then the following equalities hold
\[ y^T Ax = \lambda_A y^T x, \quad y^T Bx = \lambda_B y^T x. \]

Since \( A \leq B, \ B = A + C, \) where \( C \geq 0. \) So,
\[ \lambda_B y^T x = y^T Bx = y^T (A + C)x = y^T Ax + y^T Cx \geq y^T Ax = \lambda_A y^T x. \]

Assuming that \( y^T x > 0, \) the above relations imply that \( \lambda_B \geq \lambda_A. \) The case where \( y^T x = 0 \) is covered by using a continuity argument and perturbation technique. It is also obvious that the inequality becomes strict in the case where the associated Perron-Frobenius properties are strong.

2.2 Weyl-Tyrtyshnikov equal distribution

This part recalls some definitions on the distribution of matrix sequences. Furthermore, some tools to evaluate the strength of the equal distribution and equal localization that are based upon estimates of the singular values and involve the Frobenius norm. We denote by \( \mathcal{M}_s(\mathbb{C}) \) the linear space of all the square complex matrices of dimension \( s \times s, \) and we equip this linear space by the Frobenius norm defined by:
\[ \| A \|_F = \left( \sum_{j=1}^{s} \sigma_j(A)^2 \right)^{\frac{1}{2}} = \left( \sum_{i=1}^{s} \sum_{j=1}^{s} |a_{ij}|^2 \right)^{\frac{1}{2}} \]
where $A = [a_{ij}]_{i,j=1}^{n} \in \mathcal{M}_n(\mathbb{C})$ and $\sigma_j(A)$ denotes the $j$-th singular value of $A$. The first motivation is "practical" in the sense that, in the approximation of matrix sequences of increasing dimension in the simpler space of matrices, this is the only Shatten $p$-norm whose calculation is computationally not expensive. The second motivation is theoretical: actually the Frobenius norm is the only Shatten $p$-norm induced by an inner product which makes the space $\mathcal{M}_n(\mathbb{C})$ into a Hilbert space. More specifically, setting $\langle A, B \rangle := \text{trace}(A^* B)$, we deduce that $\|A\|_F = \langle A, A \rangle^{\frac{1}{2}}$.

**Definition 2.6.** Two real sequences $\{a_{i}^{(n)}\}_{i \leq d_n}$, $\{b_{i}^{(n)}\}_{i \leq d_n}$ ($d_n < d_{n+1}$) are equally distributed (ED) if and only if, for any real-valued continuous function $F$ with bounded support, the following relation holds:

$$
\lim_{n \to \infty} \frac{1}{d_n} \sum_{i=1}^{d_n} \left( F(a_{i}^{(n)}) - F(b_{i}^{(n)}) \right) = 0.
$$

(2.8)

When the previous limit goes to zero as $O(d_n^{-1})$ and $F$ is Lipschitz continuous, we say that there is strong equal distribution (SED). The same definition applies to the case of sequences of matrices $\{A_n\}_n$ and $\{B_n\}_n$ of dimension $d_n \times d_n$: in this case $\{a_{i}^{(n)}\}_{i \leq d_n}$ and $\{b_{i}^{(n)}\}_{i \leq d_n}$ are the sets of their singular values (or eigenvalues if the involved matrices are Hermitian).

**Notation** $\{A_n\}_n \simeq_D \{B_n\}_n$ means that the matrix sequences $\{A_n\}_n$ and $\{B_n\}_n$ are equally distributed.

**Definition 2.7.** Two real sequences $\{a_{i}^{(n)}\}_{i \leq d_n}$, $\{b_{i}^{(n)}\}_{i \leq d_n}$ ($d_n < d_{n+1}$) are equally localized (EL) if and only if, for any nontrivial interval $[\alpha, \beta]$ ($\alpha < \beta$), the following relation holds:

$$
\lim_{n \to \infty} \frac{1}{d_n} \left( \text{card}\{i : a_{i}^{(n)} \in [\alpha, \beta]\} - \text{card}\{i : b_{i}^{(n)} \in [\alpha, \beta]\} \right) = 0.
$$

(2.9)

When the previous limit goes to zero as $O(d_n^{-1})$, we say that there is strong equal localization (SEL). The same definition applies to the case of matrix sequences $\{A_n\}_n$ and $\{B_n\}_n$ of dimension $d_n \times d_n$: in this case $\{a_{i}^{(n)}\}_{i \leq d_n}$ and $\{b_{i}^{(n)}\}_{i \leq d_n}$ are the sets of their singular values (or eigenvalues if the involved matrices are Hermitian).

**Notation** $\{A_n\}_n \simeq_L \{B_n\}_n$ means that the matrix sequences $\{A_n\}_n$ and $\{B_n\}_n$ are equally localized.

**Proposition 2.7.** [9, 11, 12]. Let $\{A_n\}_n$ and $\{B_n\}_n$ be two sequences of $d_n \times d_n$ matrices.

1. Assume that $\text{rank}(A_n - B_n) = o(d_n)$. Then the sequences $\{A_n\}_n$ and $\{B_n\}_n$ are equally localized (EL) and equally distributed (ED).

2. If $\text{rank}(A_n - B_n) = O(1)$. Then the sequences $\{A_n\}_n$ and $\{B_n\}_n$ are strongly equally localized (SEL) and strongly equally distributed (SED).

**Proof.** 1. Let $r_n = \text{rank}(A_n - B_n)$. As a consequence of the Cauchy interlace theorem we have $\sigma_{i-2r_n}(B_n) \geq \sigma_{i}(A_n) \geq \sigma_{i+2r_n}(B_n)$ for $i = 2r_n + 1, \ldots, d_n - 2r_n$. Therefore, for any interval $[\alpha, \beta]$ we have

$$
\text{card}\{i : \sigma_i(A_n) \in [\alpha, \beta]\} = \text{card}\{i : \sigma_i(B_n) \in [\alpha, \beta]\} + e_n \quad |e_n| \leq 4r_n.
$$

(2.10)

Consequently $r_n = o(d_n)$ and then the sequences $\{A_n\}_n$ and $\{B_n\}_n$ are equally localized (EL). Hence, the equal distribution (since the equal localization (EL) implies the equal distribution).

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2. If \( r_n = O(1) \), then there is SEL by (2.10). For the proof of the last part, recall that \( F \) is Lipschitz continuous with bounded support contained in \( M = [\alpha, \beta] \). Owing to its Lipschitzness, \( F \) is of bounded variation \( (F \in BV) \) too. Therefore it can be expressed as the sum of two monotone functions. By linearity it is enough to focus our attention on the monotone functions restricted to \( M \). Let \( S(A_n) \) and \( S(B_n) \) be the sets of the singular values ordered nonincreasingly. Let \( q \) be an integer number and let \( S(B_n, q) \) be such that \( (S(B_n))_i = (S(B_n))_{i+q}, \quad i = 1, 2, \ldots, d_n \), where \( (S(B_n))_i = \min\{\alpha, (S(B_n))_{d_n}\} \) if \( j \geq d_n + 1 \) and \( (S(B_n))_j = \max\{\beta, (S(B_n))_1\} \) if \( j \leq 0 \). Now, supposing that \( r_n = O(1) \) i.e., \( r_n \leq k \) for some positive \( k \), we find that \( S(B_n, 2k) \geq S(B_n) \), \( S(A_n) \geq S(B_n, 2k) \), where \( " \geq " \) is intended componentwise. Finally, by monotonicity we deduce that

\[
\left| \sum_{i=1}^{d_n} (F(\sigma_i(A_n)) - F(\sigma_i(B_n))) \right| \\
\leq \sum_{i=1}^{d_n} \left| F(\sigma_i(S(B_n, -2k))) - F(\sigma_i(S(B_n, 2k))) \right| \\
= \frac{1}{d_n} \sum_{i=1-2k, \ldots, 2k; j=d_n-2k+1, \ldots, d_n+2k} (F(\sigma_i(S(B_n))) - F(\sigma_j(B_n))) \\
= O\left(d_n^{-1}\right)
\]

and the proof is complete.

\[\square\]

**Theorem 2.8.** [9, 11, 12]. Let \( \{A_n\}_n \) and \( \{B_n\}_n \) be two sequences of \( d_n \times d_n \) matrices.

1. If \( \|A_n - B_n - D_n\|_F = o(d_n) \) and \( \text{rank}(D_n) = o(d_n) \), then the sequences \( \{A_n\}_n \) and \( \{B_n\}_n \) are equally distributed (ED).

2. When \( \|A_n - B_n - D_n\|_1 = O(1) \), with \( \text{rank}(D_n) = o(1) \), then \( \{A_n\}_n \) and \( \{B_n\}_n \) are strongly equally distributed (SED).

3. **Approximation of Poisson equations (1.1) by collocation systems**

In the preceding section we have recalled some definitions and main results of the Perron-Frobenius theory and of the Weyl-Tytyshnikov equal distribution. Here, by the use of the radial function \( \phi : \mathbb{R}^n \rightarrow \mathbb{R}^+ \), we determine the collocation systems approximating the Poisson equations (1.1).

Discretizing \( \Omega \) with grid points \( z_{jk} = (x_j, y_k) = (h_j, h_k); \quad j, k = 0, 1, \ldots, n + 1 \) and \( h = \frac{1}{n+1} \), we define an approximated solution of the Poisson equations (1.1) by setting

\[
v(x, y) = \sum_{j=0}^{n+1} \sum_{k=0}^{n+1} v_{jk} \phi ((x, y) - (x_j, y_k)) \tag{3.1}
\]

where

\[
\phi(x, y) = \begin{cases} \\
\sqrt{x^2 + y^2 + c^2} & \text{Multiquadric (MQ)} \\
\frac{1}{\sqrt{x^2 + y^2 + c^2}} & \text{Inverse multiquadric (IMQ)} \\
e^{-\frac{x^2+y^2}{c^2}} & \text{Gaussian}
\end{cases} \tag{3.2}
\]

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The use of (1.1) and (3.1) yields

\[
\begin{align*}
\sum_{j=0}^{n+1} \sum_{k=0}^{n+1} v_{jk} \left[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right] (x - x_j, y - y_k) &= f(x, y) \quad \text{for } (x, y) \in \Omega \\
\sum_{j=0}^{n+1} \sum_{k=0}^{n+1} v_{jk} \phi(-x_j, y - y_k) &= g(0, y) \quad \text{if } y \in [0, 1] \\
\sum_{j=0}^{n+1} \sum_{k=0}^{n+1} v_{jk} \phi(1 - x_j, y - y_k) &= g(1, y) \quad \text{if } y \in [0, 1] \\
\sum_{j=0}^{n+1} \sum_{k=0}^{n+1} v_{jk} \phi(x - x_j, -y_k) &= g(x, 0) \quad \text{if } x \in (0, 1) \\
\sum_{j=0}^{n+1} \sum_{k=0}^{n+1} v_{jk} \phi(x - x_j, 1 - y_k) &= g(x, 1) \quad \text{if } x \in (0, 1)
\end{align*}
\]

By straightforward computations, one has:

\[
\begin{align*}
\frac{\partial \phi}{\partial x}(x, y) &= \begin{cases} 
  x(x^2 + y^2 + c^2)^{-\frac{3}{2}} \\
  -x(x^2 + y^2 + c^2)^{-\frac{3}{2}} + \frac{2}{c^2} x e^{-\frac{c^2}{2}x^2} 
\end{cases} \\
\frac{\partial \phi}{\partial y}(x, y) &= \begin{cases} 
  y(x^2 + y^2 + c^2)^{-\frac{3}{2}} \\
  -y(x^2 + y^2 + c^2)^{-\frac{3}{2}} + \frac{2}{c^2} y e^{-\frac{c^2}{2}y^2}
\end{cases}
\end{align*}
\]

then

\[
\begin{align*}
\frac{\partial^2 \phi}{\partial x^2}(x, y) &= \begin{cases} 
  (x^2 + y^2 + c^2)^{-\frac{3}{2}} - x^2(x^2 + y^2 + c^2)^{-\frac{3}{2}} \\
  -2x^2 e^{-\frac{c^2}{2}x^2} + 4x^2 e^{-\frac{c^2}{2}(x^2 + y^2 + c^2)} 
\end{cases} \\
\frac{\partial^2 \phi}{\partial y^2}(x, y) &= \begin{cases} 
  (x^2 + y^2 + c^2)^{-\frac{3}{2}} - y^2(x^2 + y^2 + c^2)^{-\frac{3}{2}} \\
  -2y^2 e^{-\frac{c^2}{2}y^2} + 4y^2 e^{-\frac{c^2}{2}(x^2 + y^2 + c^2)} 
\end{cases}
\end{align*}
\]

then

\[
\begin{align*}
\frac{\partial^2 \phi}{\partial x^2}(x, y) + \frac{\partial^2 \phi}{\partial y^2}(x, y) &= \begin{cases} 
  (x^2 + y^2 + 2c^2)(x^2 + y^2 + c^2)^{-\frac{3}{2}} \\
  (x^2 + y^2 - 2c^2)(x^2 + y^2 + c^2)^{-\frac{3}{2}} \\
  2((x^2 + y^2) - c^2) e^{-\frac{c^2}{2}x^2}
\end{cases} \\
\text{MQ} \\
\text{IMQ} \\
\text{Gaussian}
\end{align*}
\]

The linear system associated with (3.3) is given by:

\[
\begin{align*}
(a) \sum_{l,p=0}^{n+1} v_{lp} \left[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right] (x_j - x_l, y_k - y_p) &= f(x_j, y_k) : j, k = 1, \ldots, n \\
(b) \sum_{l,p=0}^{n+1} v_{lp} \phi(-x_l, y_j - y_p) &= g(0, y_j) : j = 0, 1, \ldots, n + 1 \\
(c) \sum_{l,p=0}^{n+1} v_{lp} \phi(1 - x_l, y_j - y_p) &= g(1, y_j) : j = 0, 1, \ldots, n + 1 \\
(d) \sum_{l,p=0}^{n+1} v_{lp} \phi(x_j - x_l, -y_p) &= g(x_j, 0) : j = 1, \ldots, n \\
(e) \sum_{l,p=0}^{n+1} v_{lp} \phi(x_j - x_l, 1 - y_p) &= g(x_j, 1) : j = 1, \ldots, n
\end{align*}
\]
Setting $g = \frac{c}{v}$, then

$$c_{k-p}^{j-l} = \left[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right] (x_j - x_l, y_k - y_p)$$

$$= \frac{1}{(x_j - x_l)^2 + (y_k - y_p)^2 + c^2} + \frac{c^2}{(x_j - x_l)^2 + (y_k - y_p)^2 - 2c^2}$$

$$= \frac{4}{c^2}((x_j - x_l)^2 + (y_k - y_p)^2 - c^2)e^{-\frac{(x_j - x_l)^2 + (y_k - y_p)^2}{c^2}}$$

and

$$\phi_{k-p}^{j-l} = \phi(x_j - x_l, y_k - y_p)$$

$$= \frac{1}{\pi} \frac{1}{(j-l)^2 + (k-p)^2 + g^2} + \frac{1}{\pi} \frac{1}{(j-l)^2 + (k-p)^2 + 2g^2}$$

$$= \frac{4}{\pi^2} \frac{1}{(j-l)^2 + (k-p)^2 + g^2}e^{-\frac{(j-l)^2 + (k-p)^2 + g^2}{\pi^2}}$$

It follows from (a), (b), (c), (d), and (e) that

$$\sum_{l,p=0}^{n+1} c_{k-p}^{j-l} v_{lp} = f_{jk} \quad j, k = 1, 2, \ldots, n$$ (3.8)

$$\sum_{l,p=0}^{n+1} \phi_{k-p}^{j-l} v_{lp} = g(j, y_l) = g_{0j} \quad j = 0, 1, \ldots, n + 1$$ (3.9)

$$\sum_{l,p=0}^{n+1} \phi_{j-p}^{n+1-l} v_{lp} = g(1, y_j) = g_{1j} \quad j = 0, 1, \ldots, n + 1$$ (3.10)

$$\sum_{l,p=0}^{n+1} \phi_{j-p}^{j-1} v_{lp} = g(x_j, 0) = g_{j0} \quad j = 1, \ldots, n$$ (3.11)

$$\sum_{l,p=0}^{n+1} \phi_{n+1-p}^{j-1} v_{lp} = g(x_j, 1) = g_{j1} \quad j = 1, \ldots, n$$ (3.12)

When exploiting the relations (3.8), (3.9), (3.10), (3.11), and (3.12) above we deduce the following linear system

Darbose
\[
\begin{bmatrix}
A_{0-0}^{(n+2)} & A_{0-1}^{(n+2)} & \ldots & A_{0-n}^{(n+2)} & A_{0-n-1}^{(n+2)} \\
A_{1-0}^{(n+2)} & A_{1-1}^{(n+2)} & \ldots & \ldots & A_{1-n-1}^{(n+2)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A_{n-0}^{(n+2)} & A_{n-1}^{(n+2)} & \ldots & A_{n-n}^{(n+2)} & A_{n-n-1}^{(n+2)} \\
A_{n+1-0}^{(n+2)} & A_{n+1-1}^{(n+2)} & \ldots & A_{n+1-n}^{(n+2)} & A_{n+1-n-1}^{(n+2)}
\end{bmatrix}
\begin{bmatrix}
v(0) \\
v(1) \\
\vdots \\
v(n) \\
v(n+1)
\end{bmatrix}
= \begin{bmatrix}
f(0) \\
f(1) \\
\vdots \\
f(n) \\
f(n+1)
\end{bmatrix}
\]

i.e.,

\[A_d v = f\]  

where \(d_n = (n+2)^2\). For \(j = 0, 1, \ldots, n + 1\)

\[A_{0-j}^{(n+2)} = [\phi_{1-p,l,p=0}^{j-1}; A_{n+1-j}^{(n+2)} = [\phi_{1-p,l,p=0}^{(n+1)-j,n+1}]
\]

for \(j = 1, 2, \ldots, n\) and \(l = 0, 1, \ldots, n + 1\)

\[A_j^{(n+2)} = \begin{bmatrix}
\phi_0^{j-l} & \phi_1^{j-l} & \ldots & \phi_n^{j-l} & \phi_{n+1}^{j-l} \\
\phi_0^{j-l} & \phi_1^{j-l} & \ldots & \phi_n^{j-l} & \phi_{n+1}^{j-l} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_0^{j-l} & \phi_1^{j-l} & \ldots & \phi_n^{j-l} & \phi_{n+1}^{j-l} \\
\phi_0^{j-l} & \phi_1^{j-l} & \ldots & \phi_n^{j-l} & \phi_{n+1}^{j-l}
\end{bmatrix},
\]

for \(k = 0, 1, \ldots, n + 1\),

\[v^{(k)} = (v_{k,0}, v_{k,1}, \ldots, v_{k,n+1})^T; \ f^{(0)} = (g_{0,0}, g_{0,1}, \ldots, g_{0,n+1})^T; \ f^{(n+1)} = (g_{n+1,0}, \ldots, g_{n+1,n+1})^T
\]

and for \(p = 1, 2, \ldots, n\),

\[f^{(p)} = (g_{p,0}, f_{p,1}, \ldots, f_{p,n}, g_{p,n+1})^T.
\]

The relation given by (3.13) is called the radial basis functions collocation systems approximating the elliptic boundary value problems (1.1) and the associated two level matrices \(A_d\) are the collocation matrices. These matrices are highly structured, full and near Toeplitz [3].

4. Approximation of the collocation matrices by Toeplitz sequences

This section is devoted to the approximation of collocation sequences by Toeplitz and to the asymptotic behavior of the generating function of these Toeplitz sequences.

4.1 Approximation of collocation sequences

In [1], the authors provided explicit asymptotic estimates, as function of \(c/h\), \(c\) being the shape parameter, \(h\) being the step size, to the condition number \(\mu(T_n)\) of the Toeplitz matrix \(T_n\) related to the approximated one-dimensional model problem

\[
\begin{align*}
\mu''(x) &= f(x), & x &\in (0, 1) \\
u(0) &= u_0, & u(1) &= u_1
\end{align*}
\]

Darbose

with the collocation technique over a grid of equally spaced points and based on the MQ, IMQ, and Gaussian radial functions, respectively. Here we are interested by the approximation of two-dimensional model problem (1).

In reference to section 3 and for \( l = 0, 1, \ldots, n + 1 \); if we set

\[
\Delta^{n+2}_{0\cdots l} = [\phi^{l}_{k-p} - c^{l}_{k-p}]^{n+1}_{k,p=0} \quad \text{and} \quad \Delta^{n+2}_{n+1\cdots l} = [\phi^{n+1-l}_{k-p} - c^{n+1-l}_{k-p}]^{n+1}_{k,p=0}
\]

(4.2)

then the matrices \( A^{n+2}_{0\cdots l} \), \( A^{n+2}_{n+1\cdots l} \), \( \Delta^{n+2}_{0\cdots l} \), and \( \Delta^{n+2}_{n+1\cdots l} \) are symmetric of rank at most \( n + 2 \) and the matrices \( T^{n+2}_{n+1\cdots l} \) and \( T^{n+2}_{n+1\cdots l} \) defined respectively by

\[
T^{n+2}_{0\cdots l} = A^{n+2}_{0\cdots l} - \Delta^{n+2}_{0\cdots l} \quad \text{and} \quad T^{n+2}_{n+1\cdots l} = A^{n+2}_{n+1\cdots l} - \Delta^{n+2}_{n+1\cdots l}
\]

(4.3)

are symmetric Toeplitz matrices generated by the function

\[
s(x, y) = c^{0}_{0} + 2 \sum_{k=1}^{\infty} c^{2}_{k} \cos(2k\pi x) \cos(2k\pi y) + 4 \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} c^{2}_{k} \cos(2j\pi x) \cos(2j\pi y) \quad \forall (x, y) \in \Omega
\]

(4.4)

Next, for \( j = 1, 2, \ldots, n \) and \( l = 0, 1, \ldots, n + 1 \), if we set

\[
\Delta^{n+2}_{j\cdots l} = \begin{bmatrix}
\phi^{j-l}_{0} - c^{j-l}_{0} & \cdots & \phi^{j-l}_{n+1} - c^{j-l}_{n+1} \\
0 & \ddots & 0 \\
0 & \ddots & 0 \\
\phi^{j-l}_{n+1} - c^{j-l}_{n+1} & \cdots & \phi^{j-l}_{0} - c^{j-l}_{0}
\end{bmatrix}
\] and \( T^{n+2}_{j\cdots l} = [c^{j-l}_{k-p}]^{n+1}_{k,p=0} \)

(4.5)

we deduce the following splitting

\[
A^{n+2}_{j\cdots l} = T^{n+2}_{j\cdots l} + \Delta^{n+2}_{j\cdots l}
\]

(4.6)

It follows from (4.5) that each matrix \( \Delta^{n+2}_{j\cdots l} \) is of rank equal to 2 and according to relation (4.6), it is obvious that \( \text{rank}(A^{n+2}_{j\cdots l} - T^{n+2}_{j\cdots l}) = o(n + 2) \). Exploiting Proposition 2.7 it follows that the matrix sequences \( \{A^{n+2}_{j\cdots l}\}_{n} \) and \( \{T^{n+2}_{j\cdots l}\}_{n} \) are equally distributed (ED) and equally localized (EL), i.e.,

\[
\{A^{n+2}_{j\cdots l}\}_{n} \approx_{L,D} \{T^{n+2}_{j\cdots l}\}_{n}
\]

(4.7)

When constructing the two-level matrices \( A_{d_n} \) and \( \Delta_{d_n} \) of the following way:

\[
T_{d_n} = [T^{n+2}_{j\cdots l}]_{j,l=0}^{n+1} \quad \text{and} \quad \Delta_{d_n} = [\Delta^{n+2}_{j\cdots l}]_{j,l=0}^{n+1}
\]

(4.8)

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we deduce from relations (3.14), (4.2), (4.3), (4.4), (4.5), and (4.8) that

$$ A_d = T_d + \Delta_d $$

(4.9)

where $\Delta_d$ is a matrix of order $d = (n + 2)^2$ and whose the rank satisfies the inequality

$$ \text{rank}(\Delta_d) \leq 4(n + 1) = o(d) $$

(4.10)

Next, $T_d$ is a symmetric block Toeplitz matrices with symmetric Toeplitz blocks generated by the function $s(x,y)$ defined in (4.4). Using relations (4.9) and (4.10) we have that

$$ A_d - T_d = \Delta_d $$

$$ \text{rank}(\Delta_d) = o(d) $$

(4.11)

Exploiting again Proposition 2.7 it follows that the matrix sequences $\{A_d\}_n$ and $\{T_d\}_n$ are equally distributed (ED) and equally localized (EL), i.e.,

$$ \{A_d\}_n \simeq_{L.D} \{T_d\}_n $$

(4.12)

so, the Toeplitz matrices $T_d$ are good approximations for the collocation matrices $A_d$ which can also be seen as good preconditioners for $A_d$.

4.2 Asymptotic behavior of the generating function $s(x,y)$

In this subsection we study the asymptotic behavior of the generating function $s(x,y)$ of two-level block Toeplitz matrices $T_d := T_d(s)$ in the Multiquadric, Inverse Multiquadric and Gaussian cases. First, we recall the following main result for the integrable functions.

**Proposition 4.1.** [8]. Let $\{S_n\}_n$ be a sequence of quasi-uniformly distributed grid points $x_i^{(n)}$ on $I = [-\pi, \pi]$ Then, for any bounded and Riemann integrable function $g$, we have

$$ \sum_{i=0}^{n-1} g(x_i^{(n)}) = \frac{n}{2\pi} \int_{-\pi}^{\pi} g + o(n). $$

If the distribution is uniform and if $g$ is bounded and Lipschitz continuous except, at most, for a finite number of discontinuity points, then

$$ \sum_{i=0}^{n-1} g(x_i^{(n)}) = \frac{n}{2\pi} \int_{-\pi}^{\pi} g + O(1). $$

Armed with Proposition 4.1, the following Lemma holds true.

**Lemma 4.2.** The real-valued integrable function $s(x,y)$ is even and unbounded over the compact domain $\hat{\Omega} = [-1,1]^2$.

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Proof. In this proof, we treat separately the cases: Multiquadric, Inverse Multiquadric and Gaussian. For $j, k = 0, 1, \ldots, n + 1$; let us recall that: $z_{jk} = (x_j, y_k)$, $h = \frac{1}{n+1}$ and $g = \frac{k}{n}$.

- **Multiquadric case**

  \[ c_{jk}^2 = \frac{1}{h} (j^2 + k^2 + g^2)^{-\frac{3}{2}} + \frac{g^2}{h} (j^2 + k^2 + g^2)^{-\frac{1}{2}} = c_{jk}^{(1)} + c_{jk}^{(2)} \]

  where

  \[ c_{jk}^{(1)} = \frac{1}{h} (j^2 + k^2 + g^2)^{-\frac{1}{2}} \quad \text{and} \quad c_{jk}^{(2)} = \frac{g^2}{h} (j^2 + k^2 + g^2)^{-\frac{3}{2}} \]

  Since the Fourier coefficients $c_{jk}^2$ are nonnegative, to study the behavior of the series $\sum_{j=1}^{\infty} c_{j,k}^2$, we study the behavior of the sequence $\sum_{k=1}^{n+1} c_{0,k}^{(2)}$ and we conclude by exploiting the inequality

  \[ \sum_{k=1}^{n+1} c_{0,k}^{(2)} \leq c_0^2 + 4 \left( \sum_{k=1}^{n+1} c_{0,k}^{(2)} + \sum_{k=1}^{n+1} \sum_{j=1}^{n+1} c_{j,k}^2 \right) \]

  \[ (4.13) \]

  Indeed:

  \[ c_{0,k}^{(2)} = \frac{1}{h} \frac{g^2}{(k^2 + g^2)^{\frac{3}{2}}} = \frac{g^2}{c(n+1)^3} \frac{1}{\left[ \frac{k}{n+1} \right]^2 + c^2} \frac{1}{2} = \frac{c^2}{ \left[ c^2 + y_k^2 \right]^{\frac{3}{2}}} \]

  Since the function $y \mapsto \frac{c^2}{(c^2 + y^2)^{\frac{3}{2}}}$ is positive and continuous on the interval $[0, 1]$, it is Riemann integrable, so

  \[ 0 < \int_0^1 \frac{c^2}{(c^2 + y^2)^{\frac{3}{2}}} \, dy = \alpha_0 < \infty, \]

  and according to Proposition 4.1, we have that

  \[ \lim_{n \to \infty} \frac{1}{n+1} \sum_{k=1}^{n+1} \frac{c^2}{(c^2 + y_k^2)^{\frac{3}{2}}} = \int_0^1 \frac{c^2}{(c^2 + y^2)^{\frac{3}{2}}} \, dy = \alpha_0. \]

  Then, for $\epsilon = \alpha_0 / 2$, $\exists N_{\epsilon} \in \mathbb{N}$ such that

  \[ n > N_{\epsilon} \Rightarrow \left| \frac{1}{n+1} \sum_{k=1}^{n+1} \frac{c^2}{(c^2 + y_k^2)^{\frac{3}{2}}} - \alpha_0 \right| < \frac{\alpha_0}{2} \]

  \[ \Rightarrow \frac{\alpha_0}{2} (n+1) < \sum_{k=1}^{n+1} \frac{c^2}{(c^2 + y_k^2)^{\frac{3}{2}}} < \frac{3\alpha_0}{2} (n+1) \]

  Then,

  \[ \sum_{k=1}^{n+1} c_{0,k}^{(2)} \sim n+1 \]

  Darbose
so
\[
\lim_{n \to \infty} \sum_{k=1}^{n+1} c_{0,k}^{(2)} = \infty
\]  
(4.14)

From (4.4), (4.13), and (4.14) we obtain
\[
\lim_{(x,y) \to (0,0)} s(x,y) = \infty, \quad \lim_{(x,y) \to (0,1)} s(x,y) = \infty
\]
and
\[
\lim_{(x,y) \to (1,0)} s(x,y) = \infty, \quad \lim_{(x,y) \to (1,1)} s(x,y) = \infty
\]

Hence, \(s(x,y)\) is unbounded over \(\hat{\Omega}\). Since the Toeplitz matrix \(T_{d_n}(s)\) is symmetric, we deduce that the function \(s(x,y)\) is even.

- **Inverse Multiquadric case**

\[
c_j^2 = \frac{1}{h^3} \frac{j^2 + k^2 - 2g^2}{(j^2 + k^2 + g^2)^{3/2}}
\]

then
\[
c_k^0 = \frac{1}{h^3} \frac{k^2 - 2g^2}{(k^2 + g^2)^{3/2}} = \frac{-2c^2 + (\frac{k}{n+1})^2}{[c^2 + (\frac{k}{n+1})^2]^{3/2}} = \frac{-2c^2 + y^2_k}{(c^2 + y^2_k)^{3/2}}
\]

One shows as in Multiquadric case the following relations
\[
\sum_{k=1}^{n+1} c_k^0 \sim -(n + 1) \quad \text{and} \quad \lim_{n \to \infty} \sum_{k=1}^{n+1} c_k^0 = -\infty
\]  
(4.15)

Since the Fourier coefficients \(c_j^2\) of \(s(x,y)\) are all nonpositive, we have
\[
c_0^0 + 4 \left( \sum_{k=1}^{n+1} c_k^0 + \sum_{k=1}^{n+1} \sum_{j=1}^{n+1} c_k^j \right) \leq \sum_{k=1}^{n+1} c_k^0.
\]  
(4.16)

From (4.4), (4.15), and (4.16), it follows that
\[
\lim_{(x,y) \to (0,0)} s(x,y) = -\infty, \quad \lim_{(x,y) \to (0,1)} s(x,y) = -\infty
\]
and
\[
\lim_{(x,y) \to (1,0)} s(x,y) = -\infty, \quad \lim_{(x,y) \to (1,1)} s(x,y) = -\infty
\]

Hence, \(s(x,y)\) is unbounded over the domain \(\hat{\Omega}\) and because \(T_{d_n}(s)\) is symmetric, we deduce that the function \(s(x,y)\) is even.

\[\text{Darbose}\]
• Gaussian
Also in this case, one shows as in Inverse Multiquadric case the following limits
\[
\lim_{(x,y) \to (0,0)} s(x,y) = -\infty, \quad \lim_{(x,y) \to (0,1)} s(x,y) = -\infty
\]
and
\[
\lim_{(x,y) \to (1,0)} s(x,y) = -\infty, \quad \lim_{(x,y) \to (1,1)} s(x,y) = -\infty
\]
Since the Toeplitz matrix \( T_d_n(s) \) is symmetric, we conclude that the generating function \( s(x,y) \) is even.

**Remark 4.1.** The solution of the system of linear equations \( T_d_n(s)v = \tilde{f} \) provides an approximate solution of the collocation system \( A_d_nv = \tilde{f} \) which is also an approximate solution of the elliptic boundary value problems (1.1). Furthermore, the matrices \( T_d_n(s) \) are ill-conditioned for any value of \( n \). More precisely, the Euclidean condition number of \( T_d_n(s) \), as a function of the dimensions, is unbounded:
\[
\lim_{n \to \infty} k_2(T_d_n(s)) = \infty.
\]
Hence, unless some preconditioning are used, all classic iterative methods are very slow.

5. Asymptotic growth of the spectral radii of the collocation matrices

Throughout this section we prove that the spectral radii \( \rho(A_d_n) \) of the collocation matrices \( A_d_n \) grow as \( d_n \) when the shape parameter "\( c \)" is strictly greater than \( \sqrt{2} \). The proof of this result will be done in the cases: Multiquadric, Inverse Multiquadric and Gaussian.

**Lemma 5.1.** If \( c > \sqrt{2} \) then, the inequalities (5.1), (5.2), and (5.3) hold true
\[
\rho(-T_d_n) = \rho(T_d_n) \leq \rho(A_d_n) \leq \rho(T_d_n) + \rho(\Delta_d_n)
\]

**In Multiquadric case**
\[
\min_{0 \leq j,k \leq n+1} \sum_{l=0}^{n+1} c_{k-l}^{j} \leq \rho(T_d_n) \leq \max_{0 \leq j,k \leq n+1} \sum_{l=0}^{n+1} c_{k-l}^{j}
\]
and in Inverse Multiquadric and Gaussian cases
\[
\min_{0 \leq j,k \leq n+1} \sum_{l=0}^{n+1} (-c_{k-l}) \leq \rho(-T_d_n) \leq \max_{0 \leq j,k \leq n+1} \sum_{l=0}^{n+1} (-c_{k-l})
\]

**Proof.** According to (4.9)
\[
A_d_n = T_d_n + \Delta_d_n
\]
then
\[(A_{dk})^2 = (T_{da})^2 + T_{da} \Delta_{da} + \Delta_{da} T_{da} + (\Delta_{da})^2\]  
(5.4)

since
\[T_{da} = [p^j_{j-l}]_{j,l=0}^{n+1}, \text{ and } \Delta_{da} = [\Delta^j_{j-l}]_{j,l=0}^{(n+2)n+1}\]

then
\[(T_{da})^2 = \left[ \sum_{s=0}^{n+1} \Delta^j_{j-s} \Delta^j_{s-l} \right]_{j,l=0}^{n+1} ; \quad (\Delta_{da})^2 = \left[ \sum_{s=0}^{n+1} \Delta^j_{j-s} \Delta^j_{s-l} \right]_{j,l=0}^{n+1}\]

\[T_{da} \Delta_{da} = \left[ \sum_{s=0}^{n+1} \Delta^j_{j-s} \Delta^j_{s-l} \right]_{j,l=0}^{n+1} ; \quad \Delta_{da} T_{da} = \left[ \sum_{s=0}^{n+1} \Delta^j_{j-s} \Delta^j_{s-l} \right]_{j,l=0}^{n+1}\]

then
\[(A_{dk})^2 = \left[ \sum_{s=0}^{n+1} \Delta^j_{j-s} \Delta^j_{s-l} \right]_{j,l=0}^{n+1}\]
(5.5)

For \(k, p = 0, 1, ..., n + 1\)

\[(T^j_{j-s} T^j_{s-l})_{k,p} = \left[ \sum_{q=0}^{n+1} T^j_{j-s} \Delta^j_{s-l} \right]_{j,l=0}^{n+1} = \sum_{q=0}^{n+1} c^j_{k-q} c^j_{q-p},\]
(5.6)

\[(T^j_{j-s} \Delta^j_{s-l})_{k,p} = \left[ \sum_{q=0}^{n+1} T^j_{j-s} \Delta^j_{s-l} \right]_{j,l=0}^{n+1} = \sum_{q=0}^{n+1} c^j_{k-q} (c^j_{q-p} - c^j_{q-p}),\]
(5.7)

\[(\Delta^j_{j-s} T^j_{s-l})_{k,p} = \left[ \sum_{q=0}^{n+1} \Delta^j_{j-s} T^j_{s-l} \right]_{j,l=0}^{n+1} = \sum_{q=0}^{n+1} \phi^j_{k-q} (c^j_{q-p} - c^j_{q-p}),\]
(5.8)

\[(\Delta^j_{j-s} \Delta^j_{s-l})_{k,p} = \left[ \sum_{q=0}^{n+1} \Delta^j_{j-s} \Delta^j_{s-l} \right]_{j,l=0}^{n+1} = \sum_{q=0}^{n+1} \phi^j_{k-q} (c^j_{q-p} - c^j_{q-p}),\]
(5.9)

From (5.5), (5.6), (5.7), (5.8), (5.9), we have that \((A_{dk})^2 = \left[ \sum_{j,l=0}^{n+1} a^j_{k,p} \right]_{j,l=0}^{n+1}\)

where for \(j, l = 0, 1, ..., n + 1\) and \(k, p = 0, 1, ..., n + 1\)

\[a^j_{k,p} = \sum_{s=0}^{n+1} \sum_{q=0}^{n+1} \left( c^j_{k-q} c^j_{q-p} + c^j_{k-q} (c^j_{q-p} - c^j_{q-p}) + (\phi^j_{k-q} - c^j_{k-q}) c^j_{q-p} \right).\]

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• and according to Theorem 2

\[ A \]

According to Theorem 2.

Then

\[ j,l,p,k \]

For

\[ • \]

Proof.

We treat separately the cases: Multiquadric, Inverse Multiquadric and Gaussian.

Lemma 5.2. If \( c > \sqrt{2} \), then the spectral radius \( \rho(T_{d_n}) \) of the Toeplitz matrix \( T_{d_n} \) grows as \( d_n = (n+2)^2 \).

Proof. We treat separately the cases: Multiquadric, Inverse Multiquadric and Gaussian.

• Multiquadric

For \( j, l, p, k = 0, 1, ..., n + 1 \)

\[ c_{k-p}^{j-l} = \frac{1}{h} \frac{1}{\sqrt{(j-l)^2 + (k-p)^2 + g^2}} + \frac{1}{h} \frac{g^2}{\sqrt{(j-l)^2 + (k-p)^2 + g^2}} \]

Then

\[ \sum_{l=0}^{n+1} \sum_{p=0}^{n+1} c_{k-p}^{j-l} = \frac{1}{h} \sum_{l=0}^{n+1} \sum_{p=0}^{n+1} \left( \frac{1}{\sqrt{(j-l)^2 + (k-p)^2 + g^2}} + \frac{g^2}{\sqrt{(j-l)^2 + (k-p)^2 + g^2}} \right) \]

(5.11)
First of all, for \( j - l \in [-n - 1, n + 1] \) fixed, let us study the functions: \( f_{n,j-l}(x) = \frac{1}{(g^2 + x^2 + (j-l)^2)^{\frac{3}{2}}} \) and \( g_{n,j-l}(x) = \frac{g^2}{(g^2 + x^2 + (j-l)^2)^{\frac{3}{2}}} \) over the interval \([-n - 1, n + 1]\). Since \( f_{n,j-l} \) and \( g_{n,j-l} \) are even functions, the study of these functions reduces on \([0, n + 1]\). Because \( f'_{n,j-l}(x) = \frac{-x}{(g^2 + x^2 + (j-l)^2)^{\frac{5}{2}}} \) < 0 and \( g'_{n,j-l}(x) = \frac{-3g^2x}{(g^2 + x^2 + (j-l)^2)^{\frac{5}{2}}} \) < 0 for \( x \in (0, n + 1) \), then \( f_{n,j-l} \) and \( g_{n,j-l} \) are decreasing functions on \([0, n + 1]\). So, it follows that,

\[
\min_{0 \leq j, k \leq n+1} \sum_{l=0}^{n+1} \sum_{p=0}^{n+1} c_{k-p}^{j-l} = \frac{1}{h} \sum_{l=0}^{n+1} \sum_{p=0}^{n+1} \left( \frac{1}{\sqrt{l^2 + p^2 + g^2}} + \frac{g^2}{[l^2 + p^2 + g^2]^{\frac{3}{2}}} \right) \tag{5.12}
\]

For \( l, p = 0, 1, ..., n + 1 \),

\[
\frac{1}{n + 1} \frac{1}{\sqrt{2} + c^2} \leq \frac{1}{\sqrt{l^2 + p^2 + g^2}} \quad \text{and} \quad \frac{1}{n + 1} \frac{c^2}{(2 + c^2)^{\frac{3}{2}}} \leq \frac{g^2}{(l^2 + p^2 + g^2)^{\frac{3}{2}}}
\]

then

\[
\left( \frac{1}{\sqrt{2} + c^2} + \frac{c^2}{(2 + c^2)^{\frac{3}{2}}} \right) (n + 2)^2 \leq \frac{1}{h} \sum_{l=0}^{n+1} \sum_{p=0}^{n+1} \left( \frac{1}{\sqrt{l^2 + p^2 + g^2}} + \frac{g^2}{[l^2 + p^2 + g^2]^{\frac{3}{2}}} \right) \tag{5.13}
\]

On the other side, one easily shows that

\[
\max_{0 \leq j, k \leq n+1} \sum_{l=0}^{n+1} \sum_{p=0}^{n+1} c_{k-p}^{j-l} \leq 4 \sum_{p=0}^{\frac{n+1}{2}} \sum_{p=0}^{\frac{n+1}{2}} c_p^l \leq 4 \left\{ \sum_{p=1}^{\frac{n+1}{2}} c_p^0 + 2 \sum_{p=0}^{\frac{n+1}{2}} c_p^1 + \sum_{p=1}^{\frac{n+1}{2}} \sum_{p=1}^{\frac{n+1}{2}} c_p^l \right\} \leq 4 \left[ \frac{2}{c} \frac{2}{c} (n + 2) + \frac{1}{2c} (n + 2)^2 \right] \leq \frac{18}{c} (n + 2)^2 \tag{5.14}
\]

(\( \alpha \)) follows from \( c_p^l \leq \frac{2}{c} \quad \forall l, p = 0, 1, ..., \left\lfloor \frac{n}{2} \right\rfloor + 1 \) and \( \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq \frac{n+2}{2} \). It follows from (5.2), (5.12), (5.13), and (5.14) that the spectral radius of the matrix \( T_{dn} \) grows as \( d_n = (n + 2)^2 \).

### Inverse Multiquadric

For \( j, k = 0, 1, ..., n + 1 \),

\[
\sum_{l=0}^{n+1} (-c_{k-p}^{j-l}) = \frac{1}{h^3} \sum_{l=0}^{n+1} \sum_{p=0}^{n+1} \left( \frac{2g^2 - (j - l)^2 - (k - p)^2}{[(j - l)^2 + (k - p)^2 + g^2]^{\frac{3}{2}}} \right) \tag{5.15}
\]

For \( j - l \in [-n - 1, n + 1] \), the function \( f_{n,j-l}(x) = \frac{-x^2 - (j - l)^2 + 2g^2}{[x^2 + (j - l)^2 + g^2]^{\frac{3}{2}}} \) defined on the interval \([-n - 1, n + 1]\) is even, then the study of \( f_{n,j-l} \) reduces on \([0, n + 1]\). Because \( f'_{n,j-l}(x) = \frac{-x(2g^2 + x^2 + (j - l)^2)}{(g^2 + x^2 + (j - l)^2)^{\frac{5}{2}}} \) < 0,

\[ \text{Darbose} \]
0 for \( x \in (0, n+1] \) (since \( c > \sqrt{2} \)), then \( f_{n,j-l} \) is decreasing over \([0,n+1]\). Hence

\[
\min_{0 \leq j, k \leq n+1} \sum_{l=0}^{n+1} \sum_{p=0}^{n+1} (-c_{k-p}^{j-l}) = \min_{0 \leq j, k \leq n+1} \sum_{l=0}^{n+1} \sum_{p=0}^{n+1} (-c_{p}^{j-l}) \tag{5.16}
\]

Also, the function \( g_{n,p}(x) = \frac{-x^2 - 2x^2 + 2g^2}{[x^2 + p^2 + g^2]^{\frac{3}{2}}} \) is a decreasing function over the interval \([0,n+1]\). So, we deduce that

\[
\min_{0 \leq j \leq n+1} \sum_{l=0}^{n+1} \sum_{p=0}^{n+1} (-c_{p}^{j-l}) = \sum_{l=0}^{n+1} (-c_{p}^{l}) \tag{5.17}
\]

According to (5.16), (5.17), it follows that

\[
\min_{0 \leq j, k \leq n+1} \sum_{l=0}^{n+1} \sum_{p=0}^{n+1} (-c_{k-p}^{j-l}) = \sum_{l=0}^{n+1} (-c_{p}^{l}) \tag{5.18}
\]

Because, \( \min_{l} \{\min_{p}(-c_{p}^{l})\} = \frac{1}{h^3} \frac{92^2 - 2(n+1)^2}{(g^2 + 2(n+1)^2)^{\frac{3}{2}}} = \frac{1}{h^3} \frac{2}{(g^2 + 2(n+1)^2)^{\frac{3}{2}}} \), then for \( l, p = 0, 1, ..., n+1, \)

\[
\frac{1}{h^3} \frac{2}{(g^2 + 2(n+1)^2)^{\frac{3}{2}}} \leq \frac{1}{h^3} \frac{2}{g^2 + l^2 - p^2} = -c_{p}^{l}
\]

so,

\[
2 \left( \frac{c^2 - 1}{(c^2 + 2)^{\frac{3}{2}}} \right) (n+2)^2 \leq \sum_{l=0}^{n+1} (-c_{p}^{l}) \leq (n+1) \tag{5.19}
\]

Next,

\[
\max_{0 \leq j, k \leq n+1} \sum_{l=0}^{n+1} \sum_{p=0}^{n+1} (-c_{k-p}^{j-l}) \leq 4 \sum_{l=0}^{n+1} \sum_{p=0}^{n+1} (-c_{p}^{l}) \leq 4 \left\{ -c_0^l + 2 \sum_{p=1}^{[\frac{2}{3}]+1} (-c_{p}^{0}) + \sum_{l=1}^{[\frac{2}{3}]+1} \sum_{p=1}^{[\frac{2}{3}]+1} (-c_{p}^{l}) \right\} \leq 4 \left( \frac{2}{c^3} + \frac{2}{c^3 (n+2)} + \frac{1}{2c^3} (n+2)^2 \right) \leq \frac{18}{c^3} (n+2)^2 \tag{5.20}
\]

(\( \beta \)) follows from \(-c_{p}^{l} \leq 2 \frac{2}{c^3} \) \( \forall l, p = 0, 1, ..., [\frac{2}{3}]+1 \). It follows from (5.3), (5.18), (5.19), and (5.20) that the spectral radius \( \rho(T_{d_n}) \) of \( T_{d_n} \) grows as \( (n+2)^2 \).

- **Gaussian**

One shows as in Inverse Multiquadric case by considering the function \( f_{n,j-l}(x) = [g^2 - (j - l)^2 - x^2]e^{-\frac{[(j-l)x]^2}{\alpha}} \) that the spectral radius \( \rho(T_{d_n}) \) of the collocation matrix \( T_{d_n} \) grows as \( d_n = (n+2)^2 \).

\( \Box \)

Darbose
Lemma 5.3. If \( c > \sqrt{2} \), then the inequalities (5.21) hold true

\[
\rho(\Delta_{dn}) \leq 2 \max_{0 \leq j, k \leq n+1} \sum_{l=0}^{n+1} \sum_{p=0}^{n+1} (\phi_{j-k} - \phi_{j-l}) \leq 2 \sqrt{2} cd_n \tag{5.21}
\]

Proof. Since \( c > \sqrt{2} \) then all the coefficients of \( \Delta_{dn} \) are nonnegative, so \( \Delta_{dn} \) is an eventually nonnegative matrix, according to Theorem 2.3 both matrices \( \Delta_{dn} \) and \( (\Delta_{dn})^T \) possess the Perron-Frobenius property. It follows from Theorem 2.4 that

\[
\rho(\Delta_{dn}) \leq \max_{0 \leq j, k \leq n+1} \sum_{l=0}^{n+1} \sum_{p=0}^{n+1} (\phi_{j-k} - \phi_{l-p}) \leq 2 \sqrt{2} cd_n \tag{MQ}
\]

or

\[
\rho(\Delta_{dn}) \leq 2 \max_{0 \leq j, k \leq n+1} \sum_{l=0}^{n+1} \sum_{p=0}^{n+1} (\phi_{j-k} - \phi_{l-p}) \leq \frac{2 \sqrt{2} cd_n}{2d_n} \tag{IMQ}
\]

or

\[
\rho(\Delta_{dn}) \leq \frac{2 \sqrt{2} cd_n}{2d_n} \tag{Gaussian}
\]

Lemma 5.4. If \( c > \sqrt{2} \), then the spectral radius \( \rho(A_{dn}) \) of the collocation matrix \( A_{dn} \) grows as \( d_n = (n + 2)^2 \).

Proof. The proof follows from (5.1), Lemmas 5.2 and 5.3. \qed

6. Conclusion and future works

In this paper we have studied in detail the asymptotic behavior of the generating function of the block Toeplitz matrices which are equally distributed and equally localized as the collocation matrices approximating elliptic boundary value problems (1.1) and we have provided a deep analysis of the asymptotic growth of the spectral radii of collocation matrices. Our future researches will consist to solve the preconditioned collocation systems by the quasi minimal residual (QMR) method with preconditioner chosen in the Tau algebra.

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References