A vacation queue with exceptional service for the customers

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ABSTRACT

We consider an M/G/1 queueing system in which the first $N$ customers of each busy period receive exceptional service, in addition the server takes vacation each time the system becomes empty. Using supplementary variable approach, we derive the general queue length distribution at an arbitrary time, and the moments of queue length are obtained. Numerical examples are provided.

Keywords: Exceptional service; vacation time; steady state probabilities; queue length.

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1. Introduction

In this paper, we consider a single server queueing system where the service time of each customer depends on the number of customers served prior to him in the current busy period (Exceptional services) and with server vacations. Several researchers have studied queueing systems in which the service time of a customer depends on the number of customers served in the current busy period. To mention a few references we would name Welch \cite{8}, Yukata Baba \cite{9}, Li et al \cite{6} and Igaki et al \cite{3}. The single server queueing systems with vacation are analysed by many authors including Levy et al \cite{5}, Cooper \cite{1}, Doshi \cite{2} and Yukata Baba \cite{10}. Teghem \cite{7} has made a comprehensive survey of queueing system with vacation.

The model treated in this paper is immediately applicable to many fields such as computer systems, telecommunication systems and production systems. The organisation of the paper is as follows. We describe the model and introduce notation in section 2. In section 3, we obtain the steady state probabilities and the moments of the queue length distribution. The operating characteristics are obtained in section 4 and a numerical study is carried out in section 5 to test the effect of the system parameters on the performance measures.

2. The Model

Consider a single server queueing system in which arrival follows Poisson with parameter $\lambda$ and service time distribution depends on the number of customers served since the beginning of current busy period. The service discipline is FCFS. Let $B_n(x)$ denote the service time distribution function when $n$ customers have been served since the beginning of current busy period. The Laplace-Stieltjes transform (LST) of

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$B_n(x)$ is defined by $B_n^*(x) = \int_0^\infty e^{-sx} B_n(dx)$. Further more, we assume that the service time distribution becomes stable after some customers have been served in the current busy period. That is, there is a positive integer $N \geq 1$ such that $B_n(x) = B_N(x)$ for $n \geq N$. Also if there is no customer in the queue then the server is allowed to leave the service station for a random period of time whose probability distribution is $V(x)$. He repeats the vacation period until he finds at least one in the queue.

Let $L(t)$ be the number of customers in the system including the one in the server at time $t$. Let $M(t)$ be the number of customers served since the beginning of current busy period at time $t$. Let $\hat{B}(t)$ be the remaining vacation time if there are some customers in system at time $t$. If the server is on vacation at time $t$, $M(t)$ is defined to be zero. Let $\hat{V}(t)$ be the remaining vacation time for the server on vacation.

Let us define the following probabilities for our subsequent analysis.

\[
P_{i,j}(x,t)dx = P_{i}(\{L(t) = i, M(t) = j, x < \hat{B}(t) \leq x + dx\}; (i = 1, 2, \ldots & j = 0, 1, \ldots, N - 1)
\]

\[
P_{i,N}(x,t)dx = P_{i}(\{L(t) = i, M(t) \geq N, x < \hat{B}(t) \leq x + dx\}; (i = 1, 2, \ldots)
\]

\[
Q_i(x,t)dx = P_{i}(\{L(t) = i, x < \hat{V}(t) \leq x + dx\}; (i = 0, 1, 2, \ldots)
\]

\[
P_{i,j}^*(s,t) = \int_0^\infty e^{-sx} P_{i,j}(x,t)dx; \quad i = 1, 2, \ldots \text{ and } j = 0, 1, \ldots, N
\]

\[
Q_i^*(s,t) = \int_0^\infty e^{-sx} Q_i(x,t)dx; \quad i = 0, 1, \ldots
\]

\[
V^*(s) = \int_0^\infty e^{-sx} V(x)dx;
\]

The queue length distribution at an arbitrary time will be treated by the supplementary variable technique: that is, the joint distribution of the queue length, the number of customers served since the beginning of current busy period, and the remaining service time for the customer receiving service if the server is busy, or the remaining vacation time if the server is on vacation at time $t$ form a Markov chain.

Observing the system state at time $t$ and $t + \Delta t$ and taking the limit of $\Delta t \to 0$, we have the following partial differential-difference equations.

\[
\frac{\partial Q_0(x,t)}{\partial t} - \frac{\partial Q_0(x,t)}{\partial x} = -\lambda Q_0(x,t) + \sum_{j=0}^{N} P_{1,j}(0,t)V(x) + Q_0(0,t)V(x)
\]  \quad (2.1)

\[
\frac{\partial Q_i(x,t)}{\partial t} - \frac{\partial Q_i(x,t)}{\partial x} = -\lambda Q_i(x,t) + \lambda Q_{i-1}(x,t); (i = 1, 2, \ldots)
\]  \quad (2.2)

\[
\frac{\partial P_{1,0}(x,t)}{\partial t} - \frac{\partial P_{1,0}(x,t)}{\partial x} = -\lambda P_{1,0}(x,t) + Q_1(0,t) \frac{B_0(dx)}{dx}
\]  \quad (2.3)

\[
\frac{\partial P_{i,0}(x,t)}{\partial t} - \frac{\partial P_{i,0}(x,t)}{\partial x} = -\lambda P_{i,0}(x,t) + \lambda P_{i-1,0}(x,t) + Q_i(0,t) \frac{B_0(dx)}{dx},
\]  \quad (i = 2, 3, \ldots)

\[
\frac{\partial P_{i,j}(x,t)}{\partial t} - \frac{\partial P_{i,j}(x,t)}{\partial x} = -\lambda P_{i,j}(x,t) + \lambda P_{i-1,j}(0,t) \frac{B_j(dx)}{dx},
\]  \quad (j = 1, 2, \ldots, N - 1)

\[
\frac{\partial P_{i,j}(x,t)}{\partial t} - \frac{\partial P_{i,j}(x,t)}{\partial x} = -\lambda P_{i,j}(x,t) + \lambda P_{i-1,j}(x,t) + P_{i+j-1,0}(0) \frac{B_j(dx)}{dx},
\]  \quad (2.6)
For further analysis we assume that the system is stable. We discuss the stability condition later. Let

\[ \frac{\partial P_{1,N}(x,t)}{\partial t} - \frac{\partial P_{1,N}(x,t)}{\partial x} = -\lambda P_{1,N}(x,t) + \{P_{2,N-1}(0,t) + P_{2,N}(0,t)\} \frac{B_N(dx)}{dx}, \quad (i = 2, 3, \ldots, j = 1, 2, \ldots, N-1) \]

(2.7)

\[ \frac{\partial P_{1,N}(x,t)}{\partial t} - \frac{\partial P_{1,N}(x,t)}{\partial x} = -\lambda P_{1,N}(x,t) + \lambda P_{1,N}(x,t) \]

+ \{P_{1+1,N-1}(0,t) + P_{1+1,N}(0,t)\} \frac{B_N(dx)}{dx}, \quad (i = 2, 3, \ldots) \]

(2.8)

Taking limit as \( t \to \infty \), we obtain the following equilibrium results from (1)-(8), using \( \lim_{t \to \infty} \frac{\partial P_{1,j}(x,t)}{\partial t} = 0, (i = 1, 2, \ldots ; j = 0, \ldots, N) \) and \( \lim_{t \to \infty} \frac{\partial Q_j(x,t)}{\partial t} = 0, (i = 0, 1, \ldots) \).

(2.9)

\[ -\frac{dQ_0(x)}{dx} = -\lambda Q_0(x) + \sum_{j=0}^{N} P_{1,j}(0)V(x) + Q_0(0)V(x) \]

(2.10)

\[ \frac{dQ_i(x)}{dx} = -\lambda Q_i(x) + \lambda Q_{i-1}(x); \quad (i = 1, 2, \ldots) \]

(2.11)

\[ -\frac{dP_{1,0}(x)}{dx} = -\lambda P_{1,0}(x) + Q_i(0) \frac{B_0(dx)}{dx} \]

(2.12)

\[ -\frac{dP_{1,j}(x)}{dx} = -\lambda P_{1,j}(x) + \lambda P_{1,0}(x) + Q_i(0) \frac{B_0(dx)}{dx}; \quad (i = 2, 3, \ldots) \]

(2.13)

\[ -\frac{dP_{1,j}(x)}{dx} = -\lambda P_{1,j}(x) + P_{2,j-1}(0) \frac{B_j(dx)}{dx}, \quad (j = 1, 2, \ldots, N-1) \]

(2.14)

\[ -\frac{dP_{1,1}(x)}{dx} = -\lambda P_{1,1}(x) + \{P_{2,2}(0) + P_{2,2}(0)\} \frac{B_N(dx)}{dx}, \]

(2.15)

\[ -\frac{dP_{1,N}(x)}{dx} = -\lambda P_{1,N}(x) + \lambda P_{1,N}(x) \]

+ \{P_{1+1,N-1}(0) + P_{1+1,N}(0)\} \frac{B_N(dx)}{dx}, \quad (i = 2, 3, \ldots) \]

(2.16)

Taking the LST of (9)-(16), we have

(2.17)

\[ (\lambda - s)Q_0^*(s) = \{\sum_{j=0}^{N} P_{1,j}(0) + Q_0(0)\} V^*(s) - Q_0(0), \]

(2.18)

\[ (\lambda - s)Q_i^*(s) = \lambda Q_{i-1}^*(s) - Q_i(0); \quad (i = 1, 2, \ldots) \]

(2.19)

\[ (\lambda - s)P_{1,0}^*(s) = Q_1(0)B_0^*(s) - P_{1,0}(0) \]

(2.20)

\[ (\lambda - s)P_{1,j}^*(s) = \lambda P_{1,j-1}^*(s) + Q_j(0)B_j^*(s) - P_{1,j}(0); \quad (i = 2, 3, \ldots) \]

(2.21)

\[ (\lambda - s)P_{1,j}^*(s) = 2P_1(j-1)(0)B_j^*(s) - P_{1,j}(0); \quad (j = 1, 2, \ldots, N-1) \]

\[ (\lambda - s)P_{1,N}^*(s) = \lambda P_{1,N}(s) + \lambda P_{1,N}(s) \]

+ \{P_{1+1,N-1}(0) + P_{1+1,N}(0)\} \frac{B_N(dx)}{dx}, \quad (i = 2, 3, \ldots) \]
\[(\lambda - s)P_{i,j}^* (s) = \lambda P_{i-1,j}^* (s) + P_{i+1,j-1}(0)B_j^*(s) - P_{i,j}(0), \quad (i = 2, 3, \ldots; j = 1, \ldots, N - 1)\]  

\[(\lambda - s)P_{i,N}^*(s) = \{P_{2,N-1}(0) + P_{2,N}(0)\}B_N^*(s) - P_{i,N}(0), \quad (i = 2, 3, \ldots)\]  

\[(\lambda - s)P_{i,N}^*(s) = \lambda P_{i-1,N}^*(s) + \{P_{i+1,N-1}(0) + P_{i+1,N}(0)\}B_N^*(s) - P_{i,N}(0), \quad (i = 2, 3, \ldots)\]

3. The Analysis

In this section, we obtain the expression for \(Q_{i,0}(0), P_{i,0}(0), (1 \leq i \leq N)\) and \(P_{i,j}(0)(1 \leq j \leq N)\). Substituting \(s = \lambda\) in (17) and (18), we have

\[Q_{0}(0) = \left[\sum_{j=0}^{N} P_{1,j}(0) + Q_{0}(0)\right]V^*(\lambda)\]  

\[Q_{i}(0) = \lambda Q_{i-1}^*(\lambda)\]  

Differentiating (17) and (18) \(n + 1\) times, and insertings = \(\lambda\), we have

\[-(n + 1)Q_{0}^{(n)}(\lambda) = \left[\sum_{j=0}^{N} P_{1,j}(0) + Q_{0}(0)\right]V^{(n+1)}(\lambda)\]  

\[-(n + 1)Q_{i}^{(n)}(\lambda) = \lambda Q_{i-1}^{(n+1)}(\lambda); \quad (i = 1, 2, \ldots)\]  

Using (27) and (28), we have

\[Q_{i}^*(\lambda) = \frac{(-1)^{i+1}(\lambda)^i}{(i + 1)!}\left[\sum_{j=0}^{N} P_{1,j}(0) + Q_{0}(0)\right]V^{*(i+1)}(\lambda); \quad (i = 0, 1, 2, \ldots)\]  

From (25), (26) and (29), we obtain \(Q_{i}(0)\) as

\[Q_{i}(0) = \frac{(-1)^{i}(\lambda)^i}{i!}\left[\sum_{j=0}^{N} P_{1,j}(0) + Q_{0}(0)\right]V^{*(i)}(\lambda); \quad (i = 0, 1, \ldots)\]  

Substituting \(s = \lambda\) in (19) and (20), we have

\[P_{1,0}(0) = Q_{1}(0)B_0^* (\lambda)\]  

\[P_{i,0}(0) = \lambda P_{i-1,0}^*(\lambda) + Q_{i}(0)B_0^*(\lambda); \quad (i = 2, 3, \ldots)\]  

Differentiating (19) and (20) \(n + 1\) times, and insertings = \(\lambda\), we have

\[-(n + 1)P_{1,0}^{(n)}(\lambda) = Q_{1}(0)B_0^{(n+1)}(\lambda)\]  

\[-(n + 1)P_{i,0}^{(n)}(\lambda) = \lambda P_{i-1,0}^{(n+1)}(\lambda) + Q_{i}(0)B_0^{(n+1)}(\lambda); \quad (i = 2, 3, \ldots)\]  

Using (33) and (34), we have

\[P_{i,0}^*(\lambda) = \sum_{k=1}^{i} \frac{(-1)^k(\lambda)^{k-1}}{k!}Q_{i-k+1}(0)B_0^{(k)}(\lambda); \quad (i = 1, 2, \ldots, N)\]
From (31), (32) and (35), we get $P_{i,0}(0)$ as

$$P_{i,0}(0) = \sum_{k=0}^{i-1} \frac{(-1)^k \lambda^k}{k!} Q_{i-k}(0) B_0^{(k)}(\lambda) \quad (i = 1, 2, \ldots, N) (3.12)$$

Using (30) in (36), we have

$$P_{i,0}(0) = \sum_{k=0}^{i-1} \frac{(-1)^i \lambda^i}{k!(i-k)!} \left[ \sum_{j=0}^{N} P_{i,j}(0) + Q_0(0) \right] V^{*(i-k)}(\lambda) B_0^{(k)}(\lambda) \quad (i = 1, 2, \ldots, N) (3.13)$$

Substituting $s = \lambda$ in (21) and (22), we have

$$P_{1,j}(0) = P_{2,j-1}(0) B_j^*(\lambda); \quad (j = 1, 2, \ldots, N - 1) (3.14)$$

$$P_{1,j}(0) = \lambda P_{2,j-1}(0) B_j^*(\lambda) + P_{i+1,j-1}(0) B_j^*(\lambda); \quad (i = 2, 3, \ldots and j = 1, \ldots, N - 1) (3.15)$$

Differentiating (21) and (22) $n + 1$ times, and insertings $\lambda$, we have

$$-(n + 1) P_{1,j}^{*(n)}(\lambda) = P_{2,j-1}(0) B_j^{*(n+1)}(\lambda); \quad (j = 1, 2, \ldots, N - 1) (3.16)$$

$$-(n + 1) P_{i,j}^{*(n)}(\lambda) = \lambda P_{i+1,j-1}(0) B_j^{*(n+1)}(\lambda); \quad (i = 2, 3, \ldots and j = 1, 2, \ldots, N - 1) (3.17)$$

Using (37)-(41) we can calculate $P_{i,j}(0)$ $(j = 1, 2, \ldots, N - 1; i = 1, 2, \ldots, N - j)$ in terms of $Q_0(0)$ by the following numerical algorithm.

**Algorithm**

for $j = 1$ to $N - 1$ do

for $i = 1$ to $N - j$ do

$$P_{i,j}(0) = \sum_{k=0}^{i-1} \frac{(-1)^k \lambda^k}{k!} P_{i-k+1,j-1}(0) B_j^{(k)}(\lambda) (3.18)$$

Hence $P_{i,j}(0)$ $(j = 1, 2, \ldots, N - 1)$ can be obtained from (42). Finally, from (25) we have

$$P_{i,N}(0) = \frac{1 - V^*(\lambda)}{V^*(\lambda)} Q_0(0) - \sum_{j=0}^{N-1} P_{i,j}(0) (3.19)$$

It immediately follows that we can express $P_{i,j}(0)$ $(j = 1, \ldots, N)$ in terms of $Q_0(0)$ from (37), (42) and (43).

We define the following generating functions.

$$\pi_j^*(z, s) = \sum_{i=1}^{\infty} P_{i,j}^*(s) z^i; \quad j = 0, 1, \ldots, N$$

$$q_j(z) = \sum_{i=1}^{\infty} P_{i,j}(0) z^i; \quad j = 0, 1, \ldots, N$$

$$\psi^*(z, s) = \sum_{i=0}^{\infty} Q_{i}^*(s) z^i$$

$$\phi(z) = \sum_{i=0}^{\infty} Q_{i}(0) z^i$$
Using (17)-(24) we have

\[ (\lambda - \lambda z - s)\psi^*(z, s) = \left[ \sum_{j=0}^{N} P_{1,j}(0) + Q_{0}(0) \right] V^*(s) - \phi(z) \] (3.20)

\[ (\lambda - \lambda z - s)\pi^*_0(z, s) = [\phi(z) - Q_{0}(0)]B^*_0(s) - q_0(z) \] (3.21)

\[ (\lambda - \lambda z - s)\pi^*_j(z, s) = \left[ \frac{q_{j-1}(z)}{z} - P_{1,j-1}(0) \right] B^*_j(s) - q_j(z) \] (3.22)

\[ j = 1, 2, \ldots, N - 1 \]

\[ (\lambda - \lambda z - s)\pi^*_N(z, s) = \left[ \frac{q_{N-1}(z) + q_N(z)}{z} - P_{1,N-1}(0) - P_{1,N}(0) \right] - q_N(z) \] (3.23)

Substituting \( s = \lambda - \lambda z \) in (44)-(47), we have

\[ \phi(z) = \left[ \sum_{j=0}^{N} P_{1,j}(0) + Q_{0}(0) \right] V^*(\lambda - \lambda z) \] (3.24)

\[ q_0(z) = [\phi(z) - Q_{0}(0)]B^*_0(\lambda - \lambda z) \] (3.25)

\[ q_j(z) = \left[ \frac{q_{j-1}(z)}{z} - P_{1,j-1}(0) \right] B^*_j(\lambda - \lambda z) \] (3.26)

\[ j = 1, 2, \ldots, N - 1 \]

\[ q_N(z) = \left[ \frac{q_{N-1}(z) + q_N(z)}{z} - P_{1,N-1}(0) - P_{1,N}(0) \right] B^*_N(\lambda - \lambda z) \] (3.27)

Rearranging (51), we have

\[ q_N(z) = \left[ \frac{q_{N-1}(z) - P_{1,N-1}(0)z - P_{1,N}(0)z}{z - B^*_N(\lambda - \lambda z)} \right] B^*_N(\lambda - \lambda z) \] (3.28)

Furthermore, substituting \( s = 0 \) in (44)-(47), we have.

\[ \psi^*(z, 0) = \left[ \sum_{j=0}^{N} P_{1,j}(0) + Q_{0}(0) \right] \left[ 1 - V^*(\lambda - \lambda z) \right] \] (3.29)

\[ \pi^*_0(z, 0) = \left[ \phi(z) - Q_{0}(0) \right] \left[ 1 - B^*_0(\lambda - \lambda z) \right] \] (3.30)

\[ \pi^*_j(z, 0) = \left[ \frac{q_{j-1}(z) - P_{1,j-1}(0)z}{z(\lambda - \lambda z)} \right] \left[ 1 - B^*_j(\lambda - \lambda z) \right] \] (3.31)

\[ j = 1, 2, \ldots, N - 1 \]

\[ \pi^*_N(z, 0) = \left[ \frac{q_{N-1}(z) + q_N(z) - P_{1,N-1}(0)z - P_{1,N}(0)z}{z(\lambda - \lambda z)} \right] \left[ 1 - B^*_N(\lambda - \lambda z) \right] \] (3.32)

Substituting \( z = 1 \) in (48),(49) and(50), we have

\[ \phi(1) = \sum_{j=0}^{N} P_{1,j}(0) + Q_{0}(0) \] (3.33)

\[ q_0(1) = \phi(1) - Q_{0}(0) \] (3.34)

\[ q_j(1) = q_{j-1}(1) - P_{1,j-1}(0), \quad (j = 1, \ldots, N - 1) \] (3.35)
Differentiating (48), (49) and (50), and inserting \( z = 1 \), we have

\[
\phi^{(1)}(1) = -\left[ \sum_{j=0}^{N} P_{1,j}(0) + Q_{0}(0) \right] \lambda V^{*}(1)(0) \tag{3.36}
\]

\[
q_{0}^{(1)}(1) = \phi^{(1)}(1) - [\phi(1) - Q_{0}(0)][\lambda B_{0}^{*}(1)(0)] \tag{3.37}
\]

\[
q_{j}^{(1)}(1) = q_{j-1}^{(1)}(1) - [q_{j-1}(1) - P_{1,j-1}(0)][\lambda B_{j}^{*}(1)(0)] \tag{3.38}
\]

Using L'Hospital's rule in (52), we have

\[
q_{N}(1) = \frac{q_{N-1}^{(1)}(1) - P_{1,N-1}(0) - P_{1,N}(0)}{1 + \lambda B_{N}^{*}(1)(0)} \tag{3.39}
\]

Further, using L'Hospital's rule in (53)-(56), we finally obtain,

\[
\psi^{*}(1,0) = -\left[ \sum_{j=0}^{N} P_{1,j}(0) + Q_{0}(0) \right] V^{*}(1)(0) \tag{3.40}
\]

\[
\pi_{0}^{*}(1,0) = -[\pi(1) - Q_{0}(0)] B_{0}^{*}(1)(0) \tag{3.41}
\]

\[
\pi_{j}^{*}(1,0) = -[q_{j-1}(1) - P_{1,j-1}(0)] B_{j}^{*}(1)(0); \quad (j = 1, 2, \ldots, N - 1) \tag{3.42}
\]

\[
\pi_{N}^{*}(1,0) = -[q_{N-1}(1) + q_{N}(1) - P_{1,N-1}(0) - P_{1,N}(0)] B_{N}^{*}(1)(0) \tag{3.43}
\]

Since \( P_{1,j}(0)(j = 0, \ldots, N) \) are expressed in terms of \( Q_{0}(0) \), we can express (64)-(67) also in terms of \( Q_{0}(0) \) using (57)-(67).

From the normalizing condition,

\[
\psi^{*}(1,0) + \sum_{j=0}^{N} \pi_{j}^{*}(1,0) = 1, \tag{3.44}
\]

we can find \( Q_{0}(0) \). Therefore, the steady state probabilities immediately follow.

**Remark:** It follows from (63) that this queueing system is stable if and only if \( 1 + \lambda B_{N}^{*}(1)(0) > 0 \), that is \( \lambda E(B_{N}) < 1 \).

4. **The Operating Characteristics**

In this section we derive the first and second moments of queue length. Let us define the generating function of the steady state queue length distribution as

\[
R(z) = E(z^{L}) = \psi^{*}(z, 0) + \sum_{j=0}^{N} \pi_{j}^{*}(z, 0) \tag{4.1}
\]

where \( L \) denotes the steady state queue length. Using (53)-(56) into (69), we get

\[
R(z) = E(z^{L}) \]

\[
= \frac{1}{\lambda(1 - z)} \left[ \sum_{j=0}^{N} P_{1,j}(0) + Q_{0}(0) \right] \left[ 1 - V^{*}(\lambda - \lambda z) \right]
\]
First moment of the queue length is
\[ M/G/ \]
Equation (71) is the well known generating function of the steady state queue length distribution of an
and simplifying we have
\[ B \]
in equation (42).

**Particular case (i)**

If no customer receives the exceptional service, then on setting \( B_j^*(S) = B^*(S); j = 0, 1, \ldots, N \) in (70),
and simplifying we have
\[
R(z) = \frac{B^*(\lambda - \lambda z)[V^*(\lambda - \lambda z) - 1]}{z - B^*(\lambda - \lambda z)} \left[ 1 - \lambda E(B) \right] \left[ \frac{1}{\lambda E(V)} \right]
\]

Equation (71) is the well known generating function of the steady state queue length distribution of an
\( M/G/1 \) queue with multiple vacation.

First moment of the queue length is
\[
E(L) = [R^{(1)}(z)]_{z=1} = \frac{1}{2} \phi(1) \lambda E(V^2) + \frac{1}{2} \phi(1) - Q_0(0) \lambda E(B_0^2) + \phi^{(1)}(1) E(B_0)
\]
\[
+ \sum_{j=1}^{N-1} \left[ \{q_{j-1}(1) - P_{1,j-1}(0)\} \left[ \frac{\lambda}{2} E(B_j^2) - E(B_j) \right] + \left[ q_{j-1}^{(1)}(1) - P_{1,j-1}(0)\right] E(B_j) \right]
\]
\[
+ [q_{N-1}(1) + q_N(1) - P_{1,N-1}(0) - P_{1,N}(0)] \left[ \frac{\lambda}{2} E(B_N^2) - E(B_N) \right]
\]
\[
+ \left[ q_{N-1}^{(1)}(1) + q_N^{(1)}(1) - P_{1,N-1}(0) - P_{1,N}(0)\right] E(B_N)
\]

and the second moment of the queue length is
\[
E[L(L-1)] = [R^{(2)}(z)]_{z=1} = \frac{1}{3} \phi(1) \lambda^2 E(V^3) + [\phi(1) - Q_0(0)] \lambda^2 E(B_0^2) + \phi^{(1)}(1) \lambda E(B_0^3) + \phi^{(2)}(1) E(B_0)
\]
\[
+ \sum_{j=1}^{N-1} \left[ \{q_{j-1}(1) - P_{1,j-1}(0)\} \left[ \frac{\lambda^2}{3} E(B_j^3) - \lambda E(B_j^2) + 2E(B_j) \right]
\]
\[
+ \left[ q_{j-1}^{(1)}(1) - P_{1,j-1}(0)\right] \left[ \lambda E(B_j^2) - 2E(B_j) \right] + q_{j-1}^{(2)}(1) E(B_j) \right]
\]
\[
+ [q_{N-1}(1) + q_N(1) - P_{1,N-1}(0) - P_{1,N}(0)] \left[ \frac{\lambda^2}{3} E(B_N^3) - \lambda E(B_N^2) + 2E(B_N) \right]
\]
\[
+ \left[ q_{N-1}^{(1)}(1) + q_N^{(1)}(1) - P_{1,N-1}(0) - P_{1,N}(0)\right] \left[ \lambda E(B_N^2) - 2E(B_N) \right]
\]
\[
+ \left[ q_{N-1}^{(2)}(1) + q_N^{(2)}(1)\right] E(B_N)
\]
where

\[
\phi(1) = \left[ \sum_{j=0}^N P_{1,j}(0) + Q_0(0) \right]
\]

\[
\phi^{(1)}(1) = \phi(1)\lambda E(V)
\]

\[
\phi^{(2)}(1) = \phi(1)\lambda^2 E(V)
\]

\[
\phi^{(3)}(1) = \phi(1)\lambda^3 E(V^3)
\]

\[
q_0(1) = \phi(1) - Q_0(0)
\]

\[
q_0^{(1)}(1) = [\phi(1) - Q_0(0)]\lambda E(B_0) + \phi^{(1)}(1)
\]

\[
q_0^{(2)}(1) = [\phi(1) - Q_0(0)]\lambda^2 E(B_0^2) + 2\phi^{(1)}(1)\lambda E(B_0) + \phi^{(2)}(1)
\]

\[
q_0^{(3)}(1) = [\phi(1) - Q_0(0)]\lambda^3 E(B_0^3) + 3\phi^{(1)}(1)\lambda^2 E(B_0^2) + 3\phi^{(2)}(1)\lambda E(B_0) + \phi^{(3)}(1)
\]

\[
j_1 = q_{j-1}(1) - P_{1,j-1}(0)
\]

\[
q_j^{(1)}(1) = -q_j(1) + [q_{j-1}(1) - P_{1,j-1}(0)]\lambda E(B_j) + \left[q_j^{(1)}(1) - P_{1,j-1}(0)\right]
\]

\[
q_j^{(2)}(1) = -2q_j^{(1)}(1) + [q_{j-1}(1) - P_{1,j-1}(0)]\lambda^2 E(B_j^2) + 2[q_{j-1}^{(1)}(1) - P_{1,j-1}(0)]\lambda E(B_j) + q_j^{(2)}(1)
\]

\[
q_j^{(3)}(1) = -3q_j^{(2)}(1) + [q_{j-1}(1) - P_{1,j-1}(0)]\lambda^3 E(B_j^3)
\]

\[
+ 3 \left[q_j^{(1)}(1) - P_{1,j-1}(0)\right] \lambda^2 E(B_j^2) + 3q_{j-1}^{(2)}(1)\lambda E(B_j) + q_{j-1}^{(3)}(1)
\]

\[
q_N^{(1)}(1) = \frac{1}{1 - \lambda E(B_N)} \left[q_{N-1}^{(1)}(1) - P_{1,N-1}(0) - P_{1,N}(0)\right]
\]

\[
q_N^{(1)}(1) = \frac{1}{2[1 - \lambda E(B_N)]} \lambda^2 E(B_N^2)q_{N-1}(1)
\]

\[
q_N^{(2)}(1) = \frac{1}{3[1 - \lambda E(B_N)]} \lambda^3 E(B_N^3)q_{N-1}(1) + \lambda^2 E(B_N^2)q_{N-1}(1)
\]

\[
+ 3 \left[q_{N-1}^{(1)}(1) - P_{1,N-1}(0) - P_{1,N}(0)\right] \lambda^2 E(B_N^2) + 3q_{N-1}^{(2)}(1)\lambda E(B_N) + q_{N-1}^{(3)}(1)
\]

5. A Numerical Study

In this section we numerically compare the model studied in this paper for \( N = 2 \) and the \( M/G/1 \) queue with first two customers of each busy period receiving the exceptional service without server vacation for arbitrary values of system parameters.

Let \( E(L_1) \) and \( V(L_1) \) be the mean and variance of the queue length of the current model for \( N = 2 \). Let \( E(L_2) \) and \( V(L_2) \) be the mean and variance of the queue length of \( M/G/1 \) queue with first two customers of each busy period receiving the exceptional service without server vacation. Figure 1 shows the relation between \( \lambda \) and \( E(L_1) \) and \( E(L_2) \), and figure 2 shows the variation in \( V(L_1) \) and \( V(L_2) \) with respect to \( \lambda \). From this we conclude that \( E(L_1) \) and \( V(L_1) \) are larger than \( E(L_2) \) and \( V(L_2) \).
Figure 1: $E(L_1) vs E(L_2)$

Figure 2: $V(L_1) vs V(L_2)$

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References