Application of homotopy perturbation method to nonlinear fractional population dynamics models

Zhoujin Cui, Zuodong Yang

Institute of Mathematics, School of Mathematics Science, Nanjing Normal University, Jiangsu Nanjing 210023, China

ABSTRACT

The article presents the solutions of nonlinear fractional population dynamics models with the help of analytical method of nonlinear problem called the homotopy perturbation method (HPM). The nonlinear models considered are the multi-species Lotka-Volterra equations. The numerical solutions show that only a few iterations are needed to obtain accurate approximate solutions. The method performs extremely well in terms of efficiency and simplicity to solve this model.

Keywords: Caputo fractional derivative; Lotka-Volterra model; Fractional time derivative; Homotopy perturbation method.

1. Introduction

Recently, a lot of attention has been focused on the studies of the system of nonlinear ordinary differential equations. A physical chemist by the name of Alfred James Lotka developed the notion of an evolutionary system based on two fundamental changes, those involving matter between components of a system and those involving exchanges of energy [1]. Unlike grounded in chemistry, Lotka believed that these ideas could be applied to any biological system. Around the same time as Lotka’s book: “Elements of Mathematical Biology” was published, Volterra developed the well known mathematical models of multi-species interaction. These models, the predator, prey, and competition models, are known today as Lotka-Volterra models. Although simplistic, these models are still used as the foundation for mathematical models in ecology. Biological population problems are widely investigated in many papers [2, 3]. Verhulst [4] formed a mathematical model based on “principle of population”, viz., “The logistic equation”. The first theoretical treatment of population dynamics was presented by Malthus [5], viz., “Essay on the principle of population”. The dynamic behavior of an arbitrary number of competitors can be modeled by the so-called Lotka-Volterra equations [6, 7]. These models describe the time history of a biological system and are used in such various fields as engineering, chemistry and biology. In fact, the one-predator one-prey Lotka-Volterra model is one of the most popular ones to demonstrate a simple nonlinear control system.

A general model for the dynamical system may be written as

\[
\begin{align*}
\frac{dx}{dt} &= xg(x, y), \\
\frac{dy}{dt} &= yh(x, y),
\end{align*}
\]

where \( g, h \) are arbitrary functions of the prey and predator species whose populations are \( x(t) \) and \( y(t) \) at time \( t \).
In the ecological literature, May in [8] studied the following problem

\[
\begin{cases}
\frac{dx}{dt} = ax - bxy, \\
\frac{dy}{dt} = cxy - dy,
\end{cases}
\]  

(1.2)

where \(a, b, c\) and \(d\) constants, \(x\) is the prey (victim) density, \(y\) is the predator density. May claims that system of equations 1.3 has purely neutral stability.

The last two decades have witnessed a great progress in fractional calculus and fractional-order dynamical systems. It has been found that fractional calculus is a mathematical tool that works adequately for anomalous social and physical systems with non-local, frequency- and history-dependent properties, and for intermediate states such as soft materials, which are neither idea solid nor idea fluid (see [9–12]). Differential equations with fractional-order derivatives/integrals are called fractional differential equations, and they have found many successful applications in viscoelasticity, heat conduction, electromagnetic wave, diffusion wave, control theory and so on (see [13, 14] and the references therein).

The fractional Lotka-Volterra equations are obtained from the classical equations by replacing the first order time derivatives by fractional derivatives. A significant outcome of these evolution equations is the generation of fractional Brownian motions, which are Gaussian in nature but in general non-Markovian. Fractional differential equations have garnered a lot of attention and appreciation recently due to their ability to provide an exact description of different nonlinear phenomena. The process of development of models based on fractional-order dynamical systems has lately gained popularity in the investigation of dynamical systems. The advantage of fractional-order systems is that they allow greater degrees of freedom in the model.

Except for a few special cases, it is impossible to find a closed form solution for a fractional differential equation. An effective method for solving such equations is needed. So approximate and numerical techniques must be used. The homotopy perturbation method (HPM) is relatively new approach to provide an analytical approximation to linear or nonlinear problem. This method was first presented by He [15, 16] and applied to various nonlinear problems [17, 18]. The basic difference of this method from other perturbation techniques is that it does not require small parameters in the equation, which overcomes the limitations of the traditional perturbation techniques. Recently, the application of this method is extended for fractional differential equations [19, 20].

Das et al. in [21] have solved the following fractional predator-prey model by HPM taking the constants \(a = b = c = d = 1\),

\[
\begin{cases}
D_\alpha^\alpha x = ax(t) - bx(t)y(t), \\
D_\beta^\beta y = cx(t)y(t) - dy(t).
\end{cases}
\]  

(1.3)

When \(a, b, c\) and \(d\) are functions of time \(t\), the problem of Lotka-Volterra equation with fractional time derivatives has also been solved by Das et al. in [22].

Motivated by the above results, we try to find an approximate analytical solution of multi-species Lotka-Volterra model for fractional time derivatives with the help of powerful analytical method. In this article, the problem is improved on the basis of previous work and much more realistic.

2. Fractional calculus and Basic ideas of HPM

There are several approaches to define the fractional calculus, for example, Riemann- Liouville, Grunwald-Letnikow, Caputo, and Generalized Functions approach. For the readers’ convenience, definitions of fractional integral/derivative and some preliminary results are given in this section.

**Definition 2.1** ([10]). The fractional integral of order \(\alpha \geq 0\) of a function \(x(t) : (0, +\infty) \to R\) is given by

\[
I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)ds
\]  

(2.1)
provided that the right side is point-wise defined on \((0, +\infty)\), where \(\Gamma(\cdot)\) is the well-known gamma function.

**Definition 2.2 ([10]).** The Caputo derivative of order \(s > 0\) of a continuous function \(\omega : (0, +\infty) \to \mathbb{R}\) is defined to be

\[
D^s_t \omega(t) = \frac{1}{\Gamma(n - s)} \int_0^t \frac{\omega^{(n)}(r)}{(t - r)^{n-s+1}} dr
\]

where \(n = [s] + 1\), provided that the right side is point-wise defined on \((0, +\infty)\).

**Lemma 2.1 ([10]).**

1. If \(x \in L(0,1), \rho > \sigma > 0\), then
   
   \[D^\sigma_t I^\rho_t x(t) = I^{\rho-\sigma}_t x(t),\quad I^\rho_t I^\sigma_t x(t) = I^{\rho+\sigma}_t x(t).\]

2. If \(\rho > 0, \lambda > 0\), then
   
   \[D^\rho_t t^{\lambda-1} = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\rho)} t^{\lambda-\rho-1}.\]

3. \[I^\rho_t t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\rho+\lambda+1)} t^{\lambda+\rho},\quad \rho > 0, \lambda > -1, t > 0.\]

Riemann-Liouville fractional derivative is mostly used by mathematicians but this approach is not suitable for real world physical problems since it requires the definition of fractional order initial conditions, which have no physically meaningful explanation yet. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations. We have chosen the Caputo fractional derivative because it allows traditional initial and boundary conditions to be included in the formulation of the problem. And some other properties of fractional derivative can be found in [10, 11].

To illustrate the basic ideas of HPM [15], we consider the following general nonlinear differential equation:

\[
A(y) - f(r) = 0, \quad r \in \Omega
\]

with boundary conditions

\[
B(y, \partial y / \partial n) = 0, \quad r \in \Gamma
\]

where \(A\) is a general differential operator, \(B\) is a boundary operator, \(f(r)\) is a known analytic function, and \(\Gamma\) is the boundary of the domain \(\Omega\).

The operator \(A\) can be generally divided into two parts \(L\) and \(N\), where \(L\) is linear, while \(N\) is nonlinear. Therefore the equation 2.3 can be written as follows:

\[
L(y) + N(y) - f(r) = 0
\]

We construct a homotopy of equation 2.3,

\[
y(r,p) : \Omega \times [0,1] \to \mathbb{R}
\]

which satisfies

\[
H(y,p) = (1 - p)[L(y) + L(y_0)] + p[A(y) - f(r)] = 0, \quad p \in [0,1], r \in \Omega
\]

which is equivalent to

\[
H(y,p) = L(y) - L(y_0) + pL(y_0) + p[N(y) - f(r)] = 0
\]

where \(p \in [0,1]\) is an embedding parameter, and \(y_0\) is an initial guess approximation of equation 2.3 which satisfies the boundary conditions. It follows from equations 2.6 and 2.7 that

\[
H(y,0) = L(y) - L(y_0) = 0, \quad H(y,1) = A(y) - f(r) = 0
\]
Thus, the changing process of $p$ from 0 to 1 is just that of $y(r, p)$ from $y_0(r)$ to $y(r)$. In topology this is called deformation and $L(y) - L(y_0)$ and $A(y) - f(r)$ are called homotopic. Here the embedding parameter is introduced much more naturally, unaffected by artificial factors; further it can be considered as a small parameter for $0 \leq p \leq 1$. So it is very natural to assume that the solution of the equations 2.7 and 2.8 can be expressed as

$$y(t) = y_0(t) + py_1(t) + p^2y_2(t) + \cdots$$

(2.9)

The approximate solutions of the original equations can be obtained by setting $p = 1$, that is,

$$y(t) = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n y_n(t) = y_0(t) + y_1(t) + y_2(t) + \cdots$$

(2.10)

The convergence of series (2.10) has been proved by He [15].

Because of the knowledge of various perturbation methods that low-order approximate solution leads to high accuracy, there requires no infinite series. Then after a series of recurrent calculation by using Mathematica software, we will get approximate solutions of fractional biological population model.

3. Analysis of fractional Lotka-Volterra equations

In order to assess the advantages and the accuracy of the homotopy perturbation method presented in this paper, we have applied it to the following several problems. Consider the general fractional Lotka-Volterra model for an $m$ species system given as

$$D_\alpha^i x_i = x_i (b_i + \sum_{j=1}^{m} a_{ij} x_j), \quad i = 1, 2, \ldots, m$$

(3.1)

where $0 < \alpha_i \leq 1$, the derivatives in (3.1) is the Caputo derivative. These equations may represent either predator-prey or competition cases.

3.1 One species

In this section, we will apply the HPM to solve the one species Lotka-Volterra equation. In the one-species case, equation 3.1 reduces to one species competing for a given finite source of food,

$$D_\alpha^\alpha x = x(b + ax), \quad b > 0, \quad a < 0, \quad x(0) > 0$$

(3.2)

where $0 < \alpha \leq 1$, $a$ and $b$ are constants.

To solve equation 3.2 by HPM with initial condition $x(0) = \delta$, we construct the following homotopy:

$$D_\alpha^\alpha x = p(bx + ax^2)$$

(3.3)

where the homotopy parameter $p$ is considered as a small parameter $0 \leq p \leq 1$.

Assuming the solution of equation 3.3 has the form:

$$x(t) = x_0(t) + px_1(t) + p^2x_2(t) + \cdots D_\alpha^\alpha x = p(bx + ax^2)$$

(3.4)

Substituting equation 3.4 into equation 3.3, we obtain the following set of linear differential equations:

$$p^0 : \quad D_\alpha^\alpha x_0 = 0,$$

$$p^1 : \quad D_\alpha^\alpha x_1 = bx_0 + ax_0^2,$$

$$p^2 : \quad D_\alpha^\alpha x_2 = bx_1 + 2ax_0x_1.$$
\[ p^3 : \quad D_\alpha^3 x_3 = b x_2 + a x_1^2 + 2 a x_0 x_2, \]

and so on. The method is based on applying the operators \( I_\alpha^\alpha \) (the inverse operators of the Caputo derivative \( D_\alpha^\alpha \)) on both sides of the above linear differential equations.

The first few terms of the homotopy perturbation method series for the system 3.4 are obtained as follows:

\[
\begin{align*}
x_0(t) &= \delta, \\
x_1(t) &= \frac{(b + a \delta) \delta}{\Gamma(1 + \alpha) t^\alpha}, \\
x_2(t) &= \frac{(b + a \delta)(b + 2a \delta) \delta}{\Gamma(1 + 2\alpha)} t^{2\alpha}, \\
x_3(t) &= \frac{(b + a \delta)(b + 2a \delta)^2 \delta}{\Gamma(1 + 3\alpha)} + \frac{a \delta(b + a \delta) \Gamma(1 + 2\alpha)}{\Gamma(1 + \alpha) \Gamma(1 + \alpha) \Gamma(1 + 3\alpha)} t^{3\alpha}.
\end{align*}
\]

In the similar manner the rest of the components can be obtained. Further we get the approximate solution of \( x(t) \) from 2.10.

3.2 Two species

Now we apply the HPM to solve the two-species Lotka-Volterra equation:

\[
\begin{align*}
D_\alpha^\alpha x &= x(b_1 + a_{11} x + a_{12} y), \\
D_\beta^\beta y &= y(b_2 + a_{21} x + a_{22} y)
\end{align*}
\]

where \( 0 < \alpha, \beta \leq 1, a_{11}, a_{12}, a_{21}, a_{22}, b_1 \) and \( b_2 \) are constants and subject to the initial conditions

\[
\begin{align*}
x(0) &= \delta, \\
y(0) &= \gamma
\end{align*}
\]

According to the HPM, we can construct a homotopy of system of equations 3.5 as follows:

\[
\begin{align*}
D_\alpha^\alpha x &= px(b_1 + a_{11} x + a_{12} y), \\
D_\beta^\beta y &= py(b_2 + a_{21} x + a_{22} y)
\end{align*}
\]

where the homotopy parameter \( p \) is considered as a small parameter \( 0 \leq p \leq 1 \). Assuming the solution of system 3.7 has the form:

\[
\begin{align*}
x(t) &= x_0(t) + px_1(t) + p^2 x_2(t) + \cdots \\
y(t) &= y_0(t) + py_1(t) + p^2 y_2(t) + \cdots
\end{align*}
\]

Substituting equations 3.8 and 3.9 into equation 3.7, we obtain the following set of linear differential equations:

\[
\begin{align*}
p^0 : & \quad D_\alpha^\alpha x_0 = 0, \\
& \quad D_\beta^\beta y_0 = 0, \\
p^1 : & \quad D_\alpha^\alpha x_1 = x_0(b_1 + a_{11} x_0 + a_{12} y_0), \\
& \quad D_\beta^\beta y_1 = y_0(b_2 + a_{21} x_0 + a_{22} y_0), \\
p^2 : & \quad D_\alpha^\alpha x_2 = x_0(a_{11} x_1 + a_{12} y_1) + x_1(b_1 + a_{11} x_0 + a_{12} y_0),
\end{align*}
\]
According to the HPM, we can construct a homotopy of system 3.10 as follows:

\[ D^3_t y_2 = y_0(a_21 x_1 + a_22 y_1) + y_1(b_2 + a_21 x_0 + a_22 y_0), \]

\[ p^3 : \quad D^3_t x_3 = x_0(a_{11} x_2 + a_{12} y_2) + x_1(a_{11} x_1 + a_{12} y_1) + x_2(b_1 + a_{11} x_0 + a_{12} y_0), \]

\[ D^3_t y_3 = y_0(a_{21} x_2 + a_{22} y_2) + y_1(a_{21} x_1 + a_{22} y_1) + y_2(b_2 + a_{21} x_0 + a_{22} y_0), \]

and so on.

The first few terms of the homotopy perturbation method series for the system 3.5 are obtained as follows:

\[ x_0(t) = \delta, \quad y_0(t) = \gamma, \]

\[ x_1(t) = \frac{(b_1 + a_{11} \delta + a_{12} \gamma)}{\Gamma(1 + \alpha)} t^\alpha, \quad y_1(t) = \frac{(b_2 + a_{21} \delta + a_{22} \gamma)}{\Gamma(1 + \beta)} t^\beta, \]

\[ x_2(t) = \frac{(b_1 + a_{11} \delta + a_{12} \gamma)(b_1 + 2a_{11} \delta + a_{12} \gamma)}{\Gamma(1 + 2\alpha)} t^{2\alpha} + \frac{a_{12}(b_2 + a_{21} \delta + a_{22} \gamma)}{\Gamma(\alpha + \beta + 1)} t^{\alpha + \beta}, \]

\[ y_2(t) = \frac{(b_2 + a_{21} \delta + a_{22} \gamma)(b_2 + 2a_{21} \delta + 2a_{22} \gamma)}{\Gamma(1 + 2\beta)} t^{2\beta} + \frac{a_{21}(b_1 + a_{11} \delta + a_{12} \gamma)}{\Gamma(\alpha + \beta + 1)} t^{\alpha + \beta}, \]

\[ x_3(t) = \frac{(b_1 + 2a_{11} \delta + a_{12} \gamma)^2}{\Gamma(1 + 3\alpha)} + \frac{a_{11}(b_1 + a_{11} \delta + a_{12} \gamma)}{\Gamma(1 + 2\alpha)} t^{2\alpha} + \frac{a_{12}(b_2 + a_{21} \delta + a_{22} \gamma)}{\Gamma(1 + 2\beta)} t^{\alpha + \beta} \]

\[ + \frac{\delta a_{12}(b_1 + a_{11} \delta + a_{12} \gamma)}{\Gamma(1 + 2\alpha + \beta)} t^{2\alpha + \beta} \]

\[ + \frac{\delta a_{11}(b_2 + 2a_{21} \delta + a_{22} \gamma)}{\Gamma(1 + 2\alpha + \beta)} t^{2\alpha + \beta}, \]

\[ y_3(t) = \frac{(b_2 + 2a_{21} \delta + 2a_{22} \gamma)^2}{\Gamma(1 + 3\beta)} + \frac{a_{22}(b_2 + a_{21} \delta + a_{22} \gamma)}{\Gamma(1 + 2\alpha)} t^{2\alpha} + \frac{a_{21}(b_1 + a_{11} \delta + a_{12} \gamma)}{\Gamma(1 + 2\beta)} t^{\alpha + \beta} \]

\[ + \frac{\delta a_{21}(b_2 + 2a_{21} \delta + 2a_{22} \gamma)}{\Gamma(1 + 2\alpha + \beta)} t^{2\alpha + \beta} \]

\[ + \frac{\delta a_{22}(b_1 + a_{11} \delta + a_{12} \gamma)}{\Gamma(1 + 2\alpha + \beta)} t^{2\alpha + \beta}, \]

In the similar manner the rest of the components can be obtained. Further we get the approximate solutions of \( x(t) \) and \( y(t) \) from equations 3.8 and 3.9.

### 3.3 Three species

Finally we apply the HPM to solve the three-species Lotka-Volterra equation:

\[
\begin{align*}
D^\alpha_t x &= x(1 - x - ny - nz), \\
D^\beta_t y &= y(1 - nx - y - nz), \\
D^\tau_t z &= z(1 - mx - ny - z),
\end{align*}
\]

where \( 0 < \alpha, \beta, \tau \leq 1, \) \( m, n \) are constants and subject to the initial conditions

\[ x(0) = \delta, \quad y(0) = \gamma, \quad z(0) = \sigma \]

According to the HPM, we can construct a homotopy of system 3.10 as follows:
We can construct a homotopy of system 3.10 as follows:

\[
\begin{align*}
D_t^\beta x &= px(1 - x - my - nz), \\
D_t^\gamma y &= py(1 - nx - y - mz), \\
D_t^\delta z &= pz(1 - mx - ny - z)
\end{align*}
\]  

(3.12)

where the homotopy parameter \( p \) is considered as a small parameter \( 0 \leq p \leq 1 \).

Assuming the solution of system 3.12 has the form:

\[
x(t) = x_0(t) + px_1(t) + p^2x_2(t) + \cdots
\]  

(3.13)

\[
y(t) = y_0(t) + py_1(t) + p^2y_2(t) + \cdots
\]  

(3.14)

\[
z(t) = z_0(t) + pz_1(t) + p^2z_2(t) + \cdots
\]  

(3.15)

Substituting equations 3.13, 3.14 and 3.15 into system 3.12, we obtain the following set of linear differential equations:

\[
p^0 : \quad D_t^\beta x_0 = 0, \\
& \quad D_t^\gamma y_0 = 0, \\
& \quad D_t^\delta z_0 = 0,
\]

\[
p^1 : \quad D_t^\beta x_1 = x_0(1 - x_0 - my_0 - nz_0), \\
& \quad D_t^\gamma y_1 = y_0(1 - nx_0 - y_0 - mz_0), \\
& \quad D_t^\delta z_1 = z_0(1 - mx_0 - ny_0 - z_0),
\]

\[
p^2 : \quad D_t^\beta x_2 = x_1(1 - x_0 - my_0 - nz_0) - x_0(x_1 + my_1 + nz_1), \\
& \quad D_t^\gamma y_2 = y_1(1 - nx_0 - y_0 - mz_0) - y_0(nx_1 + y_1 + mz_1), \\
& \quad D_t^\delta z_2 = z_1(1 - mx_0 - ny_0 - z_0) - z_0(mx_1 + ny_1 + z_1),
\]

\[
p^3 : \quad D_t^\beta x_3 = x_2(1 - x_0 - my_0 - nz_0) - x_0(x_2 + my_2 + nz_2) - x_1(x_1 + my_1 + nz_1), \\
& \quad D_t^\gamma y_3 = y_2(1 - nx_0 - y_0 - mz_0) - y_0(nx_2 + y_2 + mz_2) - y_1(nx_1 + y_1 + mz_1), \\
& \quad D_t^\delta z_3 = z_2(1 - mx_0 - ny_0 - z_0) - z_0(mx_2 + ny_2 + z_2) - z_1(mx_1 + ny_1 + z_1),
\]

and so on.

The first few terms of the homotopy perturbation method series for the system 3.10 are obtained as follows:

\[
x_0(t) = \delta, \quad y_0(t) = \gamma, \quad z_0(t) = \sigma, \\
x_1(t) = \frac{(1 - \delta - m\gamma - n\sigma)\delta}{\Gamma(1 + \alpha)}, \\
y_1(t) = \frac{(1 - n\delta - m\gamma - n\sigma)\gamma}{\Gamma(1 + \beta)}, \\
z_1(t) = \frac{(1 - m\delta - n\gamma - \sigma)\sigma}{\Gamma(1 + \tau)}, \\
x_2(t) = \frac{(1 - \delta - m\gamma - n\sigma)(1 - 2\delta - m\gamma - n\sigma)\delta}{\Gamma(1 + 2\alpha)} - \frac{(1 - n\delta - \gamma - m\sigma)n\delta\gamma}{\Gamma(\alpha + \beta + 1)}t^{\alpha + \beta} - \frac{(1 - m\delta - n\gamma - \sigma)n\delta\sigma}{\Gamma(\alpha + \tau + 1)}t^{\alpha + \tau},
\]
In the similar manner the rest of the components can be obtained. Further we get the approximate solutions of \( x(t) \) and \( y(t) \) from equations 3.4 and 3.5.

4. Numerical results and discussion

In this section numerical results of the Lotka-Volterra equation for one species and two species for different fractional Brownian motions and for the standard motion \( \alpha = \beta = 1 \) are calculated for various values of time \( t \). These results are presented graphically through figures.

From the figures, it is clear to see the time evolution of population density and we also know that the numerical solutions of fractional Lotka-Volterra population model is continuous with the parameter \( \alpha \) and \( \beta \). It is seen from figures 1-3, which graphically represent one species, population density increases with time \( t \). Again for two species, seen from figures 4-6, we know that population densities always increase with the spatial variables with the parameter we choose here. Analysis and results of Lotka-Volterra population system indicate that the fractional model match the anomalous biological diffusion behavior observed in the field.
5. Conclusions

In this Letter, the HPM was employed to solve a class of population dynamics models represented by the multi-species fractional Lotka-Volterra equations. Unlike the traditional methods, the solutions here are given in series form. The approximate solution to the equation was computed without any need for special transformations, linearization or discretization. It was shown that the HPM, i.e. homotopy-perturbation method, is a powerful tool for solving analytically systems of nonlinear equations.

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