Existence of solution to a nonlinear $p$-biharmonic equation

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ABSTRACT

In this note we use the Nehari manifold and fibering maps to show existence of a solution for a nonlinear $p$-biharmonic equation in a bounded smooth domain in $\mathbb{R}^N$, when $2p < N < \frac{2pq}{q-p}$.

Keywords: Boundary value problem; Nonlinear $p$-biharmonic equation; Nehari manifold; Fibering maps.

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1. Introduction

The harmonic operator is the square of the Laplace operator and it is used to model problems in engineering especially in solid and fluid mechanics. For example the boundary value problem

$$\begin{align*}
\Delta^2 u &= f(x), \quad x \in \Omega, \\
u &= \Delta u = 0, \quad x \in \partial\Omega,
\end{align*}$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^2$, models the deflection of a plate under load while the boundary of the plate is hinged. Many authors had already study this type of equation (see [1, 5, 6] and references therein).

In this note we are interested in a nonlinear situation; namely,

$$\begin{align*}
\Delta^p u &= u^{p-1}, \quad x \in \Omega, \\
u &= \Delta u = 0, \quad x \in \partial\Omega, 
\end{align*}$$

where we assume $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$. Since our results depend on the compact Sobolev embedding $W^{2,p}(\Omega) \hookrightarrow L^q(\Omega)$, we assume $q < p^* = \frac{Np}{N-2p}$. Our main result is the following

**Theorem 1.1.** Problem (P) has at least one solution.

2. Preliminaries and proof of the theorem

We set $X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ with norm $\|u\| = \left( \int_{\Omega} (\Delta u)^p \, dx \right)^{1/p}$. Since $2p < n < \frac{2pq}{q-p}$, $X$ is compactly embedded into $L^q(\Omega)$. A function $u \in X$ is called a solution of (P) if it verifies

$$\int_{\Omega} |\Delta u(x)|^{p-2} \Delta u(x) \Delta v(x) \, dx = \int_{\Omega} u(x)^{q-1} v(x) \, dx, \quad \forall v \in X$$

(2.1)
The energy functional $\Phi : X \to \mathbb{R}$, corresponding to $(P)$, is defined by:
\[ \Phi(u) = \frac{1}{p} \int_{\Omega} (\Delta u)^p \, dx - \frac{1}{q} \int_{\Omega} u^q \, dx. \]
Clearly, critical points of $\Phi$ satisfy (2.1).

A standard approach to prove existence of positive solutions of $(P)$ is to use a Mountain Pass Lemma. Here we use the fibering method which was introduced by [4] and follow the ideas of [2, 3]. To do this we need to consider the Nehari manifold $M(\Omega)$, corresponding to $\Phi$:
\[ M(\Omega) = \{ u \in X : \langle \Phi'(u), u \rangle = 0 \}, \]
where $\langle \cdot, \cdot \rangle$ stands for the duality between $X$ and $X^*$.

Thus, we obtain
\[ M(\Omega) = \left\{ u \in X : \int_{\Omega} (\Delta u)^p \, dx = \int_{\Omega} u^q \, dx \right\}. \]

Let us observe that even though $\Phi$ is unbounded from below on $X$, but it is bounded and coercive on $M(\Omega)$. Indeed, for $u \in M(\Omega)$, $\Phi(u) = \frac{1}{p} ||u||_p^p$, so $\lim_{||u||_q \to \infty} \Phi(u) = \infty$. Let us show that the zero function is an isolated member of $M(\Omega)$ in the following sense:

**Lemma 2.1.** If $u \in M(\Omega)$ is non-zero, then $||u||_q \geq \left( \frac{1}{q} \right)^{\frac{1}{p-\frac{1}{p}}}$, where $||\cdot||_q$ stands for the usual $L^q$-norm, and $S$ for the Sobolev constant in the embedding $W^{2,p}(\Omega) \hookrightarrow L^q(\Omega)$.

**Proof.** From the embedding $W^{2,p}(\Omega) \hookrightarrow L^q(\Omega)$, we infer $||u||_q \leq S ||u||$. On the other hand, for $u \in M(\Omega)$ we have $||u|| = ||u||_q^{\frac{1}{p}}$, hence $||u||_q \geq (1/S)^{\frac{1}{p-\frac{1}{p}}}$, as desired. □

Let us now introduce the fibering maps. For $u \in X$, we define $\phi_u : (0, \infty) \to \mathbb{R}$ by $\phi_u(t) = \Phi(tu)$. Observe that for $u \in X \setminus \{0\}$, $\phi(u)$ has a unique global maximizer $t > 0$, hence $\phi'_u(t) = 0$ and $\phi''_u(t) < 0$. In particular, if $u \in M(\Omega) \setminus \{0\}$, $t = 1$. Moreover, for every positive $t$, $tu \in M(\Omega) \setminus \{0\}$ if and only if $\phi'_u(t) = 0$.

**Lemma 2.2.** Suppose $\hat{u}$ is a minimizer of $\Phi$ relative to $M(\Omega) \setminus \{0\}$. Then $\hat{u}$ is a critical point of $\Phi$ relative to $X$; that is, $\Phi'(\hat{u}) = 0$.

**Proof.** From the hypothesis we derive the following equation
\[ \Phi'(\hat{u}) = \gamma \xi'(\hat{u}), \quad (2.2) \]
where $\xi(u) = ||u||^p - ||u||_q^q$ and $\gamma$ is a Lagrange multiplier. So in particular we have
\[ \langle \Phi'(\hat{u}), \hat{u} \rangle = \gamma \langle \xi'(\hat{u}), \hat{u} \rangle. \]
Thus
\[ ||\hat{u}||^p - ||\hat{u}||_q^q = \gamma \left( 2||\hat{u}||^p - 4 ||\hat{u}||_q^q \right). \]
Since $\hat{u} \in M(\Omega)$, we deduce
\[ \gamma \left( 2||\hat{u}||^p - 4 ||\hat{u}||_q^q \right) = 0. \]
We claim $\gamma = 0$. To prove the claim we use the way of contradiction. So assuming $\gamma \neq 0$, we must have $2||\hat{u}||^p - 4 ||\hat{u}||_q^q = 0$. This, in turn, implies $||\hat{u}||_q^q = 0$, since $\hat{u} \in M(\Omega)$. This is a contradiction since $\hat{u} \neq 0$. Now that $\gamma = 0$, we infer from (2.2) that $\Phi'(\hat{u}) = 0$, as desired. □

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Lemma 2.2 suggests we consider the following minimization problem

\[
\inf_{u \in M(\Omega) \setminus \{0\}} \Phi(u). 
\]

(2.3)

**Lemma 2.3.** Problem (2.3) has a solution.

**Proof.** Let us set \(\alpha \equiv \inf_{u \in M(\Omega) \setminus \{0\}} \Phi(u)\) and let \((u_n)\) be a minimizing sequence. So \(\lim_{n \to \infty} \Phi(u_n) = \alpha\).

Since \(\Phi\) is coercive on \(M(\Omega) \setminus \{0\}\), it follows that \((u_n)\) is bounded. Hence it contains a subsequence, still denoted \((u_n)\), such that \(u_n \rightharpoonup \hat{u}\) in \(X, u \to \hat{u}\) in \(L^q(\Omega)\). Since \(u_n \in M(\Omega)\) and \(\|\hat{u}\|_q = \lim_{n \to \infty} \|u_n\|_q\), it follows Lemma 1 that \(\|\hat{u}\|_q \geq \frac{1}{S} \left(\frac{p}{q}\right)^{\frac{p}{q}}\). We know from the weak lower semi-continuity of \(\|\cdot\|\) that \(\|\hat{u}\| \geq \liminf_{n \to \infty} \|u_n\|\), but we claim in fact \(\|\hat{u}\| = \liminf_{n \to \infty} \|u_n\|\).

Proceeding to derive a contradiction we assume the contrary; that is, \(\|\hat{u}\| < \liminf_{n \to \infty} \|u_n\|\). As earlier indicated there exists a unique \(\tilde{t} > 0\) such that \(\phi'_{u}(\tilde{t}) = 0\) and \(\phi''_{u}(\tilde{t}) < 0\). From \(\phi'_{u}(\tilde{t}) = 0\), we deduce \(\hat{u} \in M(\Omega) \setminus \{0\}\). On the other hand, since \(u_n \in M(\Omega) \setminus \{0\}\), \(\phi_{u_n}(t)\) attain their maximums at 2.1, so

\[
\phi_{u_n}(1) \geq \phi_{u_n}(t) \quad \forall n.
\]

Thus \(\phi_{\hat{u}}(1) \geq \phi_{u_n}(t) \quad \forall n.\)

\[
\Phi(\hat{u}) = \liminf_{n \to \infty} \Phi(u_n) = \Phi(\hat{u}),
\]

which proves that \(\hat{u}\) is a solution of (2.3), as desired.

We are now ready to prove the Theorem stated in section 1.

**Proof the theorem.** Let \(\hat{u}\) be a solution of (2.3). Lemma 2.2 implies that \(\hat{u}\) is a critical point of \(\Phi\), hence \(\Phi'(\hat{u}) = 0\), which completes the proof of the Theorem.

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