Homotopy Perturbation Method and Reduced Differential Transform Method for Solving (1+1)-Dimensional Nonlinear Boussinesq Equation

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ABSTRACT

In this paper, we will introduce the homotopy perturbation method (HPM) and the reduced differential transform method (RDTM) for solving (1+1)-dimensional nonlinear Boussinesq equation. The analytical solution of the equation have been obtained in terms of convergent series with easily computable components. The obtained results show that the proposed methods are very powerful and convenient mathematical tool for nonlinear evolution equations in science and engineering.

Keywords: Homotopy perturbation method; Reduced differential transform method; Boussinesq equations; Evolution equation.

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1. Introduction

The nonlinear evolution equations have attracted the attention of many researchers because of their wide applications in various fields such as physics, fluid mechanics, bio-mathematics, chemical physics and other areas of science and engineering. The investigation of exact solutions for the nonlinear evolution equations is a particularly hot topic. In recent years, many effective methods have been proposed for solving the nonlinear differential equations, such that exp-function method [5], sine-cosine method [17], homogenous balance method [6], Jacobi elliptic function method [4], F-expansion method [1], variational iteration method [12], homotopy perturbation method (HPM)[7-9], reduced differential transform method (RDTM) [13,14] and so on. In this paper, we employ the homotopy perturbation method and the reduced differential transform method to solve the following (1+1)-dimensional nonlinear Boussinesq equation

\[
\begin{align*}
  u_t + v_x + uu_x &= 0, \\
  v_t + (vu)_x + u_{xxx} &= 0,
\end{align*}
\]

with initial conditions

\[
\begin{align*}
  u(x, 0) &= 2x, \\
  v(x, 0) &= x^2.
\end{align*}
\]

The RDTM is presented to overcome the demerit of complex calculation of differential transform method (DTM) [2,3]. This technique doesn’t require any discretization, linearization or small perturbations and therefore it reduces significantly the numerical computation. This method is useful to obtain exact and approximate solutions.
of nonlinear differential equations.

The homotopy perturbation method was first proposed by J.Huan He in 1998, and has been shown to solve a large class of nonlinear problems effectively and accurately with approximations converging rapidly to accurate solutions. The homotopy perturbation method is in fact, a coupling of the perturbation method and the homotopy method. This method does not depend on a small parameter in the equation. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter \( p \in [0,1] \) which is considered as a small parameter. The structure of this paper is organized as follows:

In section 2, the homotopy perturbation method is described. In section 3, the reduced differential transform method is described. In section 4, we use of the proposed methods for the (1+1)-dimensional nonlinear Boussinesq equation and conclusion is given in section 6.

2. Basic Ideas of Homotopy Perturbation Method

To illustrate the homotopy perturbation method (HPM), He [10] considered the following nonlinear differential equation:

\[
A(u) - f(r) = 0, \quad r \in \Omega,
\]

with boundary conditions

\[
B(u, \frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma,
\]

where \( A \) is a general differential operator, \( B \) is a boundary operator, \( f(r) \) is a known analytical function, \( \Gamma \) is the boundary of the domain \( \Omega \) and \( \frac{\partial}{\partial n} \) denoted differential along the normal drawn outwards from \( \Omega \).

Generally speaking, the operator \( A \) can be divided into two parts \( L \) and \( N \), where \( L \) is linear, but \( N \) is nonlinear. Therefore Eq. (1) can be rewritten as follows:

\[
L(u) + N(u) - f(r) = 0.
\]

He [10,11] constructed a homotopy \( v(r,p) : \Omega \times [0,1] \rightarrow R \) which satisfies:

\[
H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0,1], \quad r \in \Omega,
\]

or equivalently,

\[
H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0.
\]

Where \( p \in [0,1] \) is an embedding parameter and \( u_0 \) is an initial approximation of Eq. (1), which satisfies the boundary conditions. Hence, it is obvious that

\[
H(v,0) = L(v) - L(u_0) = 0,
\]

\[
H(v,1) = A(v) - f(r) = 0,
\]

and the changing process of \( p \) from zero to unity is just that of \( H(v,p) \) from \( L(v) - L(u_0) \) to \( A(v) - f(r) \). In topology, this is called deformation and \( L(v) - L(u_0) \) and \( A(v) - f(r) \) are called homotopic.

Applying the perturbation technique [16] due to the fact that \( 0 \leq p \leq 1 \) can be considered as a small parameter, we can assume that the solution of Eqs. (4) and (5) can be expressed as a power series in \( p \) as follows:

\[
v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \cdots,
\]

setting \( p \rightarrow 1 \) results in the approximate solution of Eq. (1):

\[
u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \cdots.
\]
The series (9) is convergent for most cases. However the following suggestions has been made by He [11] to find the convergence rate on nonlinear operator:

1. The second derivative of \( N(v) \) with respect to \( v \) must be small because the parameter may be relatively large, i.e., \( p \to 1 \).
2. The norm of \( L^{-1} \frac{\partial N}{\partial v} \) must be smaller than one so that the series converges.

### 3. Basic of Reduced Differential Transform Method

The basic definitions of reduced differential transform method are introduced as follows:

**Definition 3.1.** If function \( u(x,t) \) is analytic and differentiated continuously with respect to time \( t \) and space \( x \) in the domain of interest, then let

\[
U_k(x) = \frac{1}{k!} \left[ \frac{\partial^k}{\partial t^k} u(x,t) \right]_{t=0} \tag{3.1}
\]

where the \( t \)-dimensional spectrum function \( U_k(x) \) is the transformed function. In this paper, the lowercase \( u(x,t) \) represent the original function while the uppercase \( U_k(x) \) stand for the transformed function.

**Definition 3.2.** The differential inverse transform of \( U_k(x) \) is defined as follows:

\[
u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k. \tag{3.2}\]

Then combining equation (10) and (11) we write

\[
u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[ \frac{\partial^k}{\partial t^k} u(x,t) \right]_{t=0} t^k.
\]

From the above definitions, it can be found that the concept of the reduced differential transform is derived from the power series expansion. The fundamental operations of reduced differential transform method are listed in Table 1 below.

### 4. Application HPM and RDTM for (1+1)-dimensional nonlinear Boussinesq equation

We consider the following (1+1)-dimensional nonlinear Boussinesq equation

\[
\begin{align*}
  u_t + v_x + uu_x &= 0, \\
  v_t + (vu)_x + u_{xxx} &= 0, \tag{4.1}
\end{align*}
\]

with initial conditions

\[
\begin{align*}
  u(x,0) &= 2x, \\
  v(x,0) &= x^2. \tag{4.2}
\end{align*}
\]

#### I. By using Homotopy Perturbation Method (HPM)

We first construct a homotopy as fallows:

\[
(1-p) \left( \frac{\partial U}{\partial t} - \frac{\partial u_0}{\partial t} \right) + p \left( \frac{\partial U}{\partial t} + \frac{\partial V}{\partial x} + U \frac{\partial U}{\partial x} \right) = 0, \tag{4.3}
\]

\[
(1-p) \left( \frac{\partial V}{\partial t} - \frac{\partial v_0}{\partial t} \right) + p \left( \frac{\partial V}{\partial t} + \frac{\partial (VU)}{\partial x} + \frac{\partial^3 U}{\partial x^3} \right) = 0, \tag{4.4}
\]
Table 1: Reduced differential transformation

<table>
<thead>
<tr>
<th>Original function</th>
<th>Transformed function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u(x, t) )</td>
<td>( U_k(x) = \frac{1}{k!} \left[ \frac{\partial^k}{\partial t^k} u(x, t) \right]_{t=0} )</td>
</tr>
<tr>
<td>( w(x, t) = u(x, t) \pm v(x, t) )</td>
<td>( W_k(x) = U_k(x) \pm V_k(x) )</td>
</tr>
<tr>
<td>( w(x, t) = \alpha u(x, t) )</td>
<td>( W_k(x) = \alpha U_k(x) ) (( \alpha ) is a constant)</td>
</tr>
<tr>
<td>( w(x, t) = x^m t^n )</td>
<td>( W_k(x) = x^m \delta(k - n), \quad \delta(k) = \begin{cases} 1, &amp; k = 0 \ 0, &amp; k \neq 0 \end{cases} )</td>
</tr>
<tr>
<td>( w(x, t) = \frac{\partial^r}{\partial t^r} u(x, t) )</td>
<td>( W_k(x) = (k + 1) \ldots (k + r) U_{k+r}(x) = \frac{(k+r)!}{k!} U_{k+r}(x) )</td>
</tr>
<tr>
<td>( w(x, t) = \frac{\partial}{\partial x} u(x, t) )</td>
<td>( W_k(x) = \frac{\partial}{\partial x} U_k(x) )</td>
</tr>
</tbody>
</table>

and the initial approximations are as follows:

\[
\begin{align*}
U_0(x, t) &= u_0(x, t) = u(x, 0) = 2x, \\
V_0(x, t) &= v_0(x, t) = v(x, 0) = x^2, \\
U_i(x, 0) &= V_i(x, 0) = 0, \\
&i = 1, 2, 3, \ldots 
\end{align*}
\]  

\( \text{(4.5)} \)

and

\[
\begin{align*}
U &= U_0 + pU_1 + p^2U_2 + \cdots, \\
V &= V_0 + pV_1 + p^2V_2 + \cdots 
\end{align*}
\]  

\( \text{(4.6)} \)

Substituting Eqs. (17) in to Eqs. (14) and (15) and arranging the coefficients of \( p \) powers, we have

\[
p^0 : \begin{cases} \\
\frac{\partial U_0(x, t)}{\partial t} = 0, \\
\frac{\partial V_0(x, t)}{\partial t} = 0, 
\end{cases}
\]  

\( \text{(4.7)} \)

\[
p^1 : \begin{cases} \\
\frac{\partial U_1(x, t)}{\partial t} + \frac{\partial U_0(x, t)}{\partial x} + U_0(x, t) \frac{\partial U_0(x, t)}{\partial x} = 0, \\
\frac{\partial V_1(x, t)}{\partial t} + \frac{\partial V_0(x, t)}{\partial x} + \frac{\partial}{\partial x} (V_0(x, t)U_0(x, t)) = 0, 
\end{cases}
\]  

\( \text{(4.8)} \)
We will obtain
We first take the differential transform of (10) by the use of Table 1 and have following equations

\[ \begin{aligned}
&\frac{\partial U_0(x,t)}{\partial t} + \frac{\partial V_0(x,t)}{\partial x} + U_0(x,t) \frac{\partial U_1(x,t)}{\partial x} + U_1(x,t) \frac{\partial U_0(x,t)}{\partial x} = 0, \\
&\frac{\partial V_0(x,t)}{\partial t} + \frac{\partial^2 U_0(x,t)}{\partial x^2} + \frac{\partial}{\partial x} (V_0(x,t) U_0(x,t) + V_1(x,t) U_0(x,t)) = 0,
\end{aligned} \]

(4.9)

\[ \vdots \]

\[ \begin{aligned}
&\frac{\partial U_1(x,t)}{\partial t} + \frac{\partial V_{j-1}(x,t)}{\partial x} + \sum_{i=0}^{j-1} U_i(x,t) \frac{\partial U_{j-i-1}(x,t)}{\partial x} = 0, \\
&\frac{\partial V_j(x,t)}{\partial t} + \frac{\partial^2 U_{j-1}(x,t)}{\partial x^2} + \frac{\partial}{\partial x} \left( \sum_{i=0}^{j-1} V_i(x,t) U_{j-i-1}(x,t) = 0 \right), \quad j = 3, 4, \ldots
\end{aligned} \]

(4.10)

Now try to obtain a solution for Eqs. (14) and (15) with initial conditions (16), subsequently solving the above equations we have:

\[ \begin{aligned}
&U_0(x,t) = 2x, \\
&V_0(x,t) = x^2,
\end{aligned} \]

(4.11)

\[ \begin{aligned}
&U_1(x,t) = -6xt, \\
&V_1(x,t) = -6x^2t,
\end{aligned} \]

(4.12)

\[ \begin{aligned}
&U_2(x,t) = 18xt^2, \\
&V_2(x,t) = 27x^2t^2,
\end{aligned} \]

(4.13)

\[ \vdots \]

\[ \begin{aligned}
&U_n(x,t) = 2x(-3t)^n, \\
&V_n(x,t) = x^2(n+1)(-3t)^n.
\end{aligned} \]

(4.14)

We will obtain

\[ \begin{aligned}
&u = \lim_{p \to 1} U = \sum_{k=0}^{\infty} U_k(x,t) = \sum_{k=0}^{\infty} \frac{2x}{(1+3t)^{k+1}}, \\
v = \lim_{p \to 1} V = \sum_{k=0}^{\infty} V_k(x,t) = \frac{2x}{(1+3t)^2}.
\end{aligned} \]

(4.15)

Which is the exact solution of Eq. (12).

II. By using Reduced Differential Transform Method (RDTM)

We first take the differential transform of (10) by the use of Table 1 and have following equations

\[ \begin{aligned}
&(k+1)U_{k+1}(x) = - \frac{\partial V_k(x)}{\partial x} - \sum_{r=0}^{k} U_r(x) \frac{\partial U_{k-r}(x)}{\partial x}, \\
&(k+1)V_{k+1}(x) = - \frac{\partial^2 U_k(x)}{\partial x^2} - \frac{\partial}{\partial x} \left( \sum_{r=0}^{k} V_r(x) U_{k-r}(x) \right),
\end{aligned} \]

(4.16)

where the t-dimensional spectrum functions \( U_k(x) \) and \( V_k(x) \) are the transformed functions. Using the initial conditions (13), we have

\[ \begin{aligned}
&U_0(x) = 2x, \\
&V_0(x) = x^2.
\end{aligned} \]

(4.17)

Now, substituting (28) into (27), we obtain the following \( U_k(x) \) and \( V_k(x) \) values successively

\[ \begin{aligned}
&U_1(x) = -6x, \quad U_2(x) = 18x, \quad \ldots, \quad U_n(x) = 2x(-3)^n, \\
&V_1(x) = -6x^2, \quad V_2(x) = 27x^2, \quad \ldots, \quad V_n(x) = x^2(n+1)(-3)^n.
\end{aligned} \]

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Finally the differential invers transform of $U_k(x)$ and $V_k(x)$ gives

$$
\begin{align*}
  u(x,t) &= \sum_{k=0}^{\infty} U_k(x) t^k = 2x - 6xt + 18x^2t^2 + \cdots + 2x(-3t)^n + \cdots = \frac{2x}{1+3t}, \\
  v(x,t) &= \sum_{k=0}^{\infty} V_k(x) t^k = x^2 - 6x^2t + 27x^2t^2 + \cdots + x^2(n+1)(-3t)^n + \cdots = \frac{x^2}{(1+3t)^2}.
\end{align*}
$$

Which is the exact solution of equation (12).

5. Conclusion

In this paper, the homotopy perturbation method and reduced differential transform method was successfully used for finding the exact solution of (1+1)-dimensional nonlinear Boussinesq equation. The results reveal that the RDTM with less and easier computations has the same results HPM. It can be concluded that the HPM and RDTM are very powerful and efficient technique in finding exact solutions for wide classes of problems.

References