The direct algebraic method to complex nonlinear partial differential equations

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ABSTRACT

By means of the two distinct methods, the direct algebraic method and the cosine method, we successfully performed an analytic study on the (2+1)-dimensional cubic nonlinear Schrödinger equation.

Keywords: Direct algebraic method; Cosine method; (2+1)-dimensional cubic nonlinear Schrödinger equation.

1. Introduction

In recent years, the investigation of exact solutions to nonlinear partial differential equations plays an important role in the nonlinear phenomena. Nonlinear phenomena appear in a wide variety of scientific applications such as plasma physics, solid state physics and fluid dynamics. In order to better understand these nonlinear phenomena, many mathematicians and physical scientists make efforts to seek more exact solutions to them. Several powerful methods have been proposed to obtain exact solutions of nonlinear partial differential equations, such as tanh-sech method [1,2,3], extended tanh method [4, 5, 6], hyperbolic function method [7], sine-cosine method [8, 9, 10], Jacobi elliptic function expansion method [11], F-expansion method [12], direct algebraic method [13,14] and first integral method [15, 16, 17, 18].

The technique we use in this paper is due to Hereman et al. [13]. By this method, solutions are developed as series in real exponential functions which physically corresponds to mixing of elementary solutions of the linear part due to nonlinearity. The method of Hereman et al. [13] falls into the category of direct solution methods for nonlinear partial differential equations. This method is currently restricted to traveling wave solutions. In addition, depending on the number of nonlinear terms in the PDE with arbitrary numerical coefficients, it is sometimes necessary to specialize to particular values of the velocity in order to find closed form solutions. On the other hand, the Hereman et al. series method does give a systematic means of developing recursion relations. Hereman et al. direct series method can be used to solve both dissipative and non dissipative equations [13]. They take solutions of the linear equation to be of the form

\[ \exp[-k(c)(x - ct)], \]

where \( k(c) \) is a function of the velocity \( c \). The velocity is assumed constant but in general is related to the wave amplitude. It is from the solutions of the linear part that the solution of the full nonlinear partial differential
equation is synthesized. With wave number \( k \), the dispersion relation \( w = k(c) \) gives the angular frequency. The aim of this paper is to find the exact soliton solutions of the \((2+1)\)-dimensional cubic nonlinear Schrödinger equation by the direct algebraic and the cosine methods.

2. **The direct algebraic method**

Consider the nonlinear PDE:

\[
F(u, u_x, u_y, u_t, u_{xx}, u_{xy}, \ldots) = 0
\]  

(2.1)

where \( u(x, y, t) \) is the solution of the Eq. (1). We use the transformations

\[
u(x, y, t) = f(\xi), \quad \xi = k x + l y + \lambda t
\]  

(2.2)

where \( k, l \) and \( \lambda \) are constants. Based on this we obtain

\[
\frac{\partial}{\partial x}(\cdot) = k \frac{d}{d\xi}(\cdot), \quad \frac{\partial}{\partial y}(\cdot) = l \frac{d}{d\xi}(\cdot), \quad \frac{\partial}{\partial t}(\cdot) = \lambda \frac{d}{d\xi}(\cdot), \quad \frac{\partial^2}{\partial x^2}(\cdot) = k^2 \frac{d^2}{d\xi^2}(\cdot), \quad \ldots
\]  

(2.3)

We use (3) to change the PDE (1) to ODE:

\[
G(f(\xi), \frac{df(\xi)}{d\xi}, \frac{d^2 f(\xi)}{d\xi^2}, \ldots) = 0.
\]  

(2.4)

Next, we apply the approach of Hereman et al. [13]. We solve the linear terms and then suppose the solution in the form

\[
f(\xi) = \sum_{n=1}^{\infty} a_n g^n(\xi),
\]  

(2.5)

where \( g(\xi) \) is a solution of linear terms and the expansion coefficients \( a_n(n = 1, 2, \ldots) \) are to be determined. To deal with the nonlinear terms, we need to apply the extension of Cauchy’s product rule for multiple series.

**Lemma 1.** (Extension of Cauchy’s product rule) If

\[
F^{(i)} = \sum_{n_1=1}^{I} a_{n_1}^{(i)}, \quad i = 1, \ldots, I,
\]  

(2.6)

represents \( I \) infinite convergent series then

\[
\prod_{i=1}^{I} F^{(i)} = \sum_{n=1}^{\infty} \sum_{r=1}^{I-1} \ldots \sum_{m=2}^{k-1} \sum_{l=1}^{m-1} a_1^{(1)} a_2^{(2)} \ldots a_{n-r}^{(l)}.
\]  

(2.7)

**Proof.** See [13].

Substituting (5) into (4) yields recursion relation which gives the values of the coefficients.

3. **(2+1)-dimensional cubic nonlinear Schrödinger equation**

Let us consider the \((2+1)\)-dimensional cubic nonlinear Schrödinger equation [19]:

\[
i\Psi_t + c(\Psi_{xx} + \Psi_{yy}) + a|\Psi|^2\Psi + b\Psi = 0,
\]  

(3.1)
where \(a, b, c\) are non-zero constants and \(\Psi = \Psi(x, y, t)\) is a complex-valued function of three real variables \(x, y, t\).

We introduce the transformations (See Re. [15] and Re. [18])

\[
\Psi(x, y, t) = e^{i \theta} \Psi(\xi), \quad \theta = \alpha x + \beta y + \delta t, \quad \xi = kx + ly - 2c(\alpha k + \beta l)t,
\]

(3.2)

where \(k, l, \alpha, \beta\) and \(\delta\) are constants and \(\Psi(\xi)\) is real function.

Substituting (9) into (8), we obtain ordinary differential equation:

\[
c(k^2 + l^2)\Psi''(\xi) - (c(\alpha^2 + \beta^2) + \delta + b)\Psi(\xi) + a(\Psi(\xi))^3 = 0.
\]

(3.3)

### 3.1. The direct algebraic method

The linear equation from (10) has the solution in the form

\[
g(\xi) = \exp\left(\frac{c(\alpha^2 + \beta^2) + \delta + b}{c(k^2 + l^2)} \xi\right).
\]

Thus, we look for the solution of (10) in the form

\[
\Psi(\xi) = \sum_{n=1}^{\infty} a_n \exp\left(n \sqrt{\frac{c(\alpha^2 + \beta^2) + \delta + b}{c(k^2 + l^2)}} \xi\right).
\]

(3.4)

Substituting (11) into (10) and by using Lemma 1, we obtain the recursion relation follows:

\[
(c(\alpha^2 + \beta^2) + \delta + b)(n^2 - 1)a_n + a \sum_{m=2}^{n-1} \sum_{l=1}^{n-1} a_m a_{n-m} = 0, \quad n \geq 3.
\]

(3.5)

Then by (12), we have \(a_{2d} = 0\),

\[
a_{2d+1} = (-1)^d \left(\frac{a}{c(\alpha^2 + \beta^2) + \delta + b}\right)^d \frac{a_{2d+1}}{2^{2d}}, \quad d = 1, 2, 3, \ldots
\]

(3.6)

Substituting (13) into (11) gives

\[
\Psi(\xi) = \frac{8(c(\alpha^2 + \beta^2) + \delta + b)a_1 \exp\left(\sqrt{\frac{c(\alpha^2 + \beta^2) + \delta + b}{c(k^2 + l^2)}} \xi\right)}{8(c(\alpha^2 + \beta^2) + \delta + b) + aa_1^2 \exp(2\sqrt{\frac{c(\alpha^2 + \beta^2) + \delta + b}{c(k^2 + l^2)}} \xi)}.
\]

(3.7)

In \((x, y, t)\)-variables, we have the exact soliton solution of the (2+1)-dimensional cubic nonlinear Schrödinger equation in the following form

\[
\Psi = e^{i(\alpha x + \beta y + \delta t)} \frac{8(c(\alpha^2 + \beta^2) + \delta + b)a_1 \exp\left(\sqrt{\frac{c(\alpha^2 + \beta^2) + \delta + b}{c(k^2 + l^2)}} (kx + ly - 2c(\alpha k + \beta l)t)\right)}{8(c(\alpha^2 + \beta^2) + \delta + b) + aa_1^2 \exp(2\sqrt{\frac{c(\alpha^2 + \beta^2) + \delta + b}{c(k^2 + l^2)}} (kx + ly - 2c(\alpha k + \beta l)t))}.
\]

(3.8)
In (15) if we choose \( a_1 = \pm 2 \sqrt{\frac{2(c(\alpha^2 + \beta^2) + \delta + b)}{a}} \), the exact solution of the (2+1)-dimensional cubic nonlinear Schrödinger equation can be expressed as:

\[
\Psi = \pm e^{i(\alpha x + \beta y + \delta t)} \sqrt{\frac{2(c(\alpha^2 + \beta^2) + \delta + b)}{a}} \text{sech}\left[ \sqrt{\frac{c(\alpha^2 + \beta^2) + \delta + b}{c(k^2 + l^2)}}(kx + ly - 2c(\alpha k + \beta l)t) \right].
\]

(3.9)

3.2. Using the cosine method

In this section, the cosine method [8 - 10] is applied to the (2+1)-dimensional cubic nonlinear Schrödinger equation.

The solution of Eq. (10) can be expressed in the form:

\[
\Psi(\xi) = \lambda \cos^3(\mu \xi), \quad |\xi| \leq \frac{\pi}{2\mu}.
\]

(3.10)

where \( \lambda, \beta \) and \( \mu \) are unknown parameters which will be determined. Then we have:

\[
\Psi'(\xi) = -\lambda \beta \mu \cos^3(\mu \xi) \sin(\mu \xi),
\]

\[
\Psi''(\xi) = -\lambda \mu^2 \beta^2 \cos^{3} (\mu \xi) + \lambda \mu^2 \beta (\beta - 1) \cos^{3-2} (\mu \xi).
\]

(3.11)

Substituting Eq. (17) and Eq. (18) into Eq. (10) gives

\[
c(k^2 + l^2)[-\lambda \mu^2 \beta^2 \cos^3(\mu \xi) + \lambda \mu^2 \beta (\beta - 1) \cos^{3-2}(\mu \xi)]
\]

\[
-(c(\alpha^2 + \beta^2) + \delta + b) \lambda \cos^3(\mu \xi) + a \lambda^3 \cos^3 \beta (\mu \xi) = 0.
\]

(3.12)

By equating the exponents and the coefficients of each pair of the cosine function we obtain the following system of algebraic equations:

\[
c(k^2 + l^2)\lambda \mu^2 \beta (\beta - 1) + a \lambda^3 = 0,
\]

\[
-c(k^2 + l^2)\lambda \mu^2 \beta^2 - (c(\alpha^2 + \beta^2) + \delta + b)\lambda = 0,
\]

\[
3\beta = \beta - 2.
\]

(3.13)

Solving the system (20), we have

\[
\lambda = \pm \sqrt{\frac{2(c(\alpha^2 + \beta^2) + \delta + b)}{a}}, \quad \beta = -1, \quad \mu = \pm \sqrt{\frac{-c(\alpha^2 + \beta^2) + \delta + b}{c(k^2 + l^2)}}.
\]

(3.14)

Combining (21) with (17), we obtain the exact solution to equation (10) as follows:

\[
\Psi(\xi) = \pm \sqrt{\frac{2(c(\alpha^2 + \beta^2) + \delta + b)}{a}} \sec(\sqrt{\frac{-c(\alpha^2 + \beta^2) + \delta + b}{c(k^2 + l^2)}} \xi).
\]

(3.15)

Then the exact solution to the (2+1)-dimensional cubic nonlinear Schrödinger equation can be written as

\[
\Psi = \pm e^{i(\alpha x + \beta y + \delta t)} \sqrt{\frac{2(c(\alpha^2 + \beta^2) + \delta + b)}{a}} \text{sech}\left[ \sqrt{\frac{c(\alpha^2 + \beta^2) + \delta + b}{c(k^2 + l^2)}}(kx + ly - 2c(\alpha k + \beta l)t) \right].
\]

(3.16)
References