Eigenvalue Problems for System of Third Order Four-Point Nonlinear Boundary Value Problems on Time Scales

S. Nageswara Rao\(^a\), A. Kameswara Rao\(^b\)

\(^a\)Department of Mathematics, Aditya Institute of Technology and Management, Tekkali, 532 201, India
\(^b\)Department of Mathematics, Gayatri Vidya Parishad College of Engineering for Women Madhurawada, Visakhapatnam, 530 048, India

ABSTRACT

Values of \(\lambda\) are determined for which there exist positive solutions of the system of four-point nonlinear boundary value problems,

\[
\begin{align*}
\Delta^3 u(t) + \lambda p(t)f(v) &= 0, \quad t \in [a, b], \\
\Delta^3 v(t) + \lambda q(t)g(u) &= 0, \quad t \in [a, b],
\end{align*}
\]

with the boundary conditions,

\[
\begin{align*}
&u(a) = u(\xi) = 0 = \beta u(\eta) - \alpha u(\sigma^3(b)), \\
&v(a) = v(\xi) = 0 = \beta v(\eta) - \alpha v(\sigma^3(b)),
\end{align*}
\]

where \(a < \xi < \eta < \sigma^3(b), \alpha > 0, \beta > 0\).

Keywords: Time scales, Boundary value problem, System of equations, Positive solutions, Eigenvalue intervals, Cone.

1. Introduction

Let \(\mathbb{T}\) be a time scale with \(a, \sigma^3(b) \in \mathbb{T}\). Given an interval \(J\) of \(\mathbb{R}\), we will use the interval notation \(J_{\mathbb{T}} = J \cap \mathbb{T}\). We are concerned with determining values of \(\lambda\)(eigenvalues) for which there exist positive solutions for the system of four point boundary value problems

\[
\begin{align*}
\Delta^3 u(t) + \lambda p(t)f(v) &= 0, \quad t \in [a, b], \\
\Delta^3 v(t) + \lambda q(t)g(u) &= 0, \quad t \in [a, b],
\end{align*}
\]

\[(1.1)\]

\[
\begin{align*}
u(a) = u(\xi) = 0 = \beta u(\eta) - \alpha u(\sigma^3(b)), \\
v(a) = v(\xi) = 0 = \beta v(\eta) - \alpha v(\sigma^3(b)),
\end{align*}
\]

\[(1.2)\]

where \(a < \xi < \eta < \sigma^3(b), \alpha > 0, \beta > 0\).

The theory of dynamic equations on time scales(more generally, on measure chains) was introduced in 1998 by Stefan Hilger in his PhD thesis \([11]\). The theory presents a structure where, once a result is established for a general time scale, then special cases can be obtained by taking the particular time
scales. If $\mathbb{T} = \mathbb{R}$, then we have the result for differential equations. Choosing $\mathbb{T} = \mathbb{Z}$ we immediately get the result for difference equations. A great deal of work has been done since 1988, unifying and extending the theories of differential and difference equations, and many results are now available in the general setting of time scales and reference therein. For an excellent introduction to the overall area of dynamic equations on time scales, we refer to the textbook by Bohner and Peterson [4].

On a larger scale, there has been a great deal of study focused on positive solutions of boundary value problems for ordinary differential equations. Interest in such solutions is high from a theoretical sense [12] and as applications for which only positive solutions are meaningful [1]. These considerations are cast primarily for scalar problems, but good attention has been given to boundary value problems for systems of differential equations [15, 16]. Recently Bai and Gu [5], the existence of positive solutions was studied for the second order four-point boundary value problem,

$$x''(t) + \lambda h(t)f(t, x(t)) = 0, \quad 0 < t < T$$

$$x(0) = \alpha x(\xi), \quad x(1) = \beta x(\eta).$$

Moreover Benchohra et al. [6] and Henderson and Ntouyas [9] studied the existence of positive solutions of systems of nonlinear eigenvalue problems. Also, Henderson and Ntouyas [10] dealt with the existence of positive solutions of systems of nonlinear eigenvalue problems for three point boundary conditions. In this paper, we employ the methods used in some of the previous papers to extend those results to eigenvalue problems for systems of four-point boundary value problems on time scales (1.1)-(1.2).

We use the following notation for the convenience,

$$A = \beta \eta - \alpha \sigma^3(b), \quad B = \beta \xi^2 - \alpha (\sigma^3(b))^2,$$

$$k_1 = \frac{a^2(\beta - \alpha) - B}{2(A - \xi(\beta - \alpha))}, \quad k_2 = \frac{a^2(\beta - \alpha) - B}{2(a(\beta - \alpha) - A)}$$

and

$$D = A(a^2 - \xi^2) - (\beta - \alpha)a^2 \xi - a \xi^2 + B(\xi - a).$$

We make the following assumptions throughout:

(A1) $f, g \in C([0, \infty), [0, \infty))$,

(A2) $p, q \in C([a, \sigma(b)]_{\mathbb{T}}, [0, \infty))$, and each does not vanish identically on any closed subinterval of $[a, \sigma(b)]_{\mathbb{T}}$,

(A3) $\xi < \beta \eta + \alpha \sigma(b), \beta > \alpha > 0$,

(A4) all of $f_0 := \lim_{x \to 0^+} (f(x)/x), g_0 := \lim_{x \to 0^+} (g(x)/x)$,

$\quad f_\infty := \lim_{x \to \infty} (f(x)/x)$, and $g_\infty := \lim_{x \to \infty} (g(x)/x)$ exist as positive real numbers.

The main tool in this paper is an application of the Guo-Krasnosel’skii fixed point theorem for operators leaving a Banach space cone invariant [7]. A Green function plays a fundamental role in defining an appropriate operator on a suitable cone.

2. Green’s Function and Bounds

In this section, we construct the Green’s function for the homogeneous problem corresponding to the BVP (1.1)-(1.2) in three different intervals in twelve different cases and we estimate the bounds for the Green’s function.
Let $G(t, s)$ be the Green’s function for the BVP $-u^{\Delta\Delta}(t) = 0$ satisfying (1.2). After computation, the Green’s function $G(t, s)$ can be obtained as

$$
G(t, s) = \begin{cases} 
G_{11}(t, s), & a \leq \sigma(s) < t \leq \xi < \eta < \sigma^3(b), \\
G_{12}(t, s), & a \leq t < s \leq \xi < \eta < \sigma^3(b), \\
G_{13}(t, s), & a \leq t \leq \xi < s < \eta < \sigma^3(b), \\
G_{14}(t, s), & a \leq t \leq \xi < \eta < s < \sigma^3(b), \\
G_{21}(t, s), & a < \sigma(s) < \xi < t < \eta < \sigma^3(b), \\
G_{22}(t, s), & a < \xi \leq \sigma(s) < t < \eta < \sigma^3(b), \\
G_{23}(t, s), & a < \xi < s < \eta < \sigma^3(b), \\
G_{24}(t, s), & a < \xi < \eta < s < \sigma^3(b), \\
G_{31}(t, s), & a < \sigma(s) < \xi < \eta \leq t \leq \sigma^3(b), \\
G_{32}(t, s), & a < \xi < \sigma(s) < \eta \leq t \leq \sigma^3(b), \\
G_{33}(t, s), & a < \xi < \eta \leq \sigma(s) < t \leq \sigma^3(b), \\
G_{34}(t, s), & a < \xi < \eta \leq t < \sigma^3(b), \\
\end{cases}
$$

(2.1)
Theorem 2.1. Let \( G(t,s) \) be the Green’s function for the homogeneous problem \(-u^{\Delta\Delta\Delta}(t) = 0\) satisfying the boundary conditions (1.2). Then the inequality
\[
\gamma G(\sigma(s), s) \leq G(t,s) \leq G(\sigma(s), s), \quad \text{for all} \quad (t,s) \in [a, \sigma^3(b)]_T \times [a,b],
\] (2.2)
where
\[0 < \gamma = \min \left\{ \frac{G_{11}(k_1, s)}{G_{11}(\sigma(s), s)}, \frac{G_{13}(\frac{a}{2}, s)}{G_{13}(\sigma(s), s)}, \frac{G_{14}(\frac{a}{2}, s)}{G_{14}(\sigma(s), s)}, \frac{G_{12}(k_2, s)}{G_{12}(\sigma(s), s)}, \frac{G_{22}(k_2, s)}{G_{22}(\sigma(s), s)}, \frac{G_{33}(\frac{a}{2}, s)}{G_{33}(\sigma(s), s)}} \right\}. \tag{2.3}
\]

Proof. The Green’s function \(G(t, s)\) is given in (2.1) in twelve different cases. In each case we prove the inequality as in (2.2). Clearly
\[G(t, s) > 0 \quad \text{on} \quad [a, \sigma^3(b)] \times [a, b]. \tag{2.4}\]

**Case (i).** For \(a \leq \sigma(s) < t \leq \xi < \eta < \sigma^3(b)\)
\[\frac{G(t, s)}{G(\sigma(s), s)} = \frac{G_{11}(t, s)}{G_{11}(\sigma(s), s)} = \frac{[(A\xi^2 - B\xi) - t(\xi^2(\beta - \alpha) - B) - t^2(\sigma(\beta - \alpha))]}{[(A\xi^2 - B\xi) - \sigma(s)(\xi^2(\beta - \alpha) - B) - (\sigma(s))^2(\sigma(\beta - \alpha))]},\]
from (A3), we have \(G_{11}(t, s) \leq G_{11}(\sigma(s), s)\). Therefore,
\[G(t, s) \leq G(\sigma(s), s).\]

And also, from (A3), we have
\[\frac{G(t, s)}{G(\sigma(s), s)} = \frac{G_{11}(t, s)}{G_{11}(\sigma(s), s)} \geq \frac{G_{11}(k_1, s)}{G_{11}(\sigma(s), s)}.\]
Therefore,
\[G(t, s) \geq \frac{G_{11}(k_1, s)}{G_{11}(\sigma(s), s)} G(\sigma(s), s).\]

**Case (ii).** For \(a \leq t \leq \xi < s < \eta < \sigma^3(b)\)
\[\frac{G(t, s)}{G(\sigma(s), s)} = \frac{G_{13}(t, s)}{G_{13}(\sigma(s), s)} = \frac{[(a\xi^2 - a^2\xi) + t(a^2 - \xi^2) + t^2(\xi - a)]}{[(a\xi^2 - a^2\xi) + \sigma(s)(a^2 - \xi^2) + (\sigma(s))^2(\xi - a)]},\]
from (A3), we have \(G_{13}(t, s) \leq G_{13}(\sigma(s), s)\). Therefore,
\[G(t, s) \leq G(\sigma(s), s).\]

And also, from (A3), we have
\[\frac{G(t, s)}{G(\sigma(s), s)} = \frac{G_{13}(t, s)}{G_{13}(\sigma(s), s)} \geq \frac{G_{13}(\frac{a}{2}, s)}{G_{13}(\sigma(s), s)}.\]
Therefore,
\[G(t, s) \geq \frac{G_{13}(\frac{a}{2}, s)}{G_{13}(\sigma(s), s)} G(\sigma(s), s).\]
Case (iii). For \( a \leq t \leq \xi < \eta < s < \sigma^3(b) \)

\[
\frac{G(t, s)}{G(\sigma(s), s)} = \frac{G_{14}(t, s)}{G_{14}(\sigma(s), s)}
= \frac{[-(a\xi^2 - a^2\xi) - t(a^2 - \xi^2) - t^2(\xi - a)]}{[-(a\xi^2 - a^2\xi) - \sigma(s)(a^2 - \xi^2) - (\sigma(s))^2(\xi - a)]},
\]

from (A3), we have \( G_{14}(t, s) \leq G_{14}(\sigma(s), s) \). Therefore,

\[
G(t, s) \leq G(\sigma(s), s).
\]

And also, from (A3), we have

\[
\frac{G(t, s)}{G(\sigma(s), s)} = \frac{G_{14}(t, s)}{G_{14}(\sigma(s), s)} \leq \frac{G_{14}(\frac{t}{\xi}, s)}{G_{14}(\sigma(s), s)}.
\]

Therefore,

\[
G(t, s) \geq \frac{G_{14}(\frac{t}{\xi}, s)}{G_{14}(\sigma(s), s)} G(\sigma(s), s).
\]

Case (iv). For \( a \leq t < s \leq \xi < \eta < \sigma^3(b) \)

From (A3) and case (ii), we have \( G_{12}(t, s) \leq G_{12}(\sigma(s), s) \). Therefore,

\[
G(t, s) \leq G(\sigma(s), s).
\]

And also, from (A3), we have

\[
\frac{G(t, s)}{G(\sigma(s), s)} \geq \min \left\{ \frac{G_{12}(k_2, s)}{G_{12}(\sigma(s), s)}, \frac{G_{13}(\frac{t}{\xi}, s)}{G_{13}(\sigma(s), s)} \right\},
\]

Therefore,

\[
G(t, s) \geq \min \left\{ \frac{G_{12}(k_2, s)}{G_{12}(\sigma(s), s)}, \frac{G_{13}(\frac{t}{\xi}, s)}{G_{13}(\sigma(s), s)} \right\} G(\sigma(s), s).
\]

Case (v). For \( a < \xi \leq \sigma(s) < t < \eta < \sigma^3(b) \)

From (A3) and case (i), we have \( G_{22}(t, s) \leq G_{22}(\sigma(s), s) \). Therefore,

\[
G(t, s) \leq G(\sigma(s), s).
\]

And also, from (A3), we have

\[
\frac{G(t, s)}{G(\sigma(s), s)} \geq \min \left\{ \frac{G_{11}(k_1, s)}{G_{11}(\sigma(s), s)}, \frac{G_{22}(k_2, s)}{G_{22}(\sigma(s), s)} \right\},
\]

Therefore,

\[
G(t, s) \geq \min \left\{ \frac{G_{11}(k_1, s)}{G_{11}(\sigma(s), s)}, \frac{G_{22}(k_2, s)}{G_{22}(\sigma(s), s)} \right\} G(\sigma(s), s).
\]

Case (vi). For \( a < \xi < \eta < \sigma(s) < t \leq \sigma^3(b) \)

From (A3) and case (v), we have \( G_{33}(t, s) \leq G_{33}(\sigma(s), s) \). Therefore,

\[
G(t, s) \leq G(\sigma(s), s).
\]
And also, from (A3), we have
\[
\frac{G(t, s)}{G(\sigma(s), s)} \geq \min \left\{ \frac{G_{11}(k_1, s)}{G_{11}(\sigma(s), s)}, \frac{G_{22}(k_2, s)}{G_{22}(\sigma(s), s)}, \frac{G_{33}(k_3, s)}{G_{33}(\sigma(s), s)} \right\}.
\]
Therefore,
\[
G(t, s) \geq \min \left\{ \frac{G_{11}(k_1, s)}{G_{11}(\sigma(s), s)}, \frac{G_{22}(k_2, s)}{G_{22}(\sigma(s), s)}, \frac{G_{33}(k_3, s)}{G_{33}(\sigma(s), s)} \right\} G(\sigma(s), s).
\]
In other cases, the inequality can be established similarly and so their arguments omitted.

By consolidating all the above cases, we get
\[
\gamma G(\sigma(s), s) \leq G(t, s) \leq G(\sigma(s), s), \quad \text{for all } (t, s) \in [a, \sigma^3(b)]T \times [a, b],
\]
where \(\gamma\) given in (2.3).

We note that a pair \((u(t), v(t))\) is a solution of the eigenvalue problem (1.1)-(1.2) if and only if
\[
\begin{align*}
u(t) &= \lambda \int_a^{\sigma(b)} G(t, s) p(s) f(\lambda \int_a^{\sigma(s)} G(\sigma(s), r) q(r) g(u(\sigma(r))) \Delta r) \Delta s, \quad a \leq t \leq \sigma^3(b), \\
v(t) &= \lambda \int_a^{\sigma(b)} G(t, s) q(s) g(u(\sigma(s))) \Delta s, \quad a \leq t \leq \sigma^3(b).
\end{align*}
\] (2.5)

Values of \(\lambda\) for which there are positive solutions (positive with respect to a cone) of (1.1)-(1.2) will be determined via applications of the following fixed point-theorem [14].

**Theorem 2.2. (Krasnosel’skii)** Let \(B\) be a Banach space, and let \(\mathcal{P} \subset B\) be a cone in \(B\). Assume that \(\Omega_1\) and \(\Omega_2\) are open subsets of \(B\) with \(0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2\), and let \(T : \mathcal{P} \cap (\bar{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}\) be a completely continuous operator such that either

(i) \(\|Tu\| \leq \|u\|, \quad u \in \mathcal{P} \cap \partial \Omega_1, \quad \text{and} \quad \|Tu\| \geq \|u\|, \quad u \in \mathcal{P} \cap \partial \Omega_2, \quad \text{or}
\)

(ii) \(\|Tu\| \geq \|u\|, \quad u \in \mathcal{P} \cap \partial \Omega_1, \quad \text{and} \quad \|Tu\| \leq \|u\|, \quad u \in \mathcal{P} \cap \partial \Omega_2.\)

3. **Positive Solutions in a Cone**

In this section, we apply Theorem 2.2 to obtain solutions in a cone (i.e., positive solutions) of (1.1)-(1.2).

For our construction, \(\mathcal{B} = \{ x : [a, \sigma^3(b)]T \to \mathbb{R} \}\), with supremum norm, \(\|x\| = \sup \{ |x(t)| : t \in [a, \sigma^3(b)]T \}\), and define a cone \(\mathcal{P} \subset \mathcal{B}\) by
\[
\mathcal{P} = \left\{ x \in \mathcal{B} : x(t) \geq 0 \text{ on } [a, \sigma^3(b)]T \text{ and } \min_{t \in [\xi, \eta]} x(t) \geq \gamma \|x\\| \right\}.
\] (3.1)

For our first result, define positive numbers \(L_1\) and \(L_2\) by
\[
L_1 := \max \left\{ \left[ \gamma \int_{\xi}^{\eta} G(\tau, s) p(s) \Delta s \|u\|_\infty \right]^{-1}, \left[ \gamma \int_{\xi}^{\eta} G(\tau, s) q(s) \Delta s \|u\|_\infty \right]^{-1} \right\}
\]
\[
L_2 := \min \left\{ \left[ \gamma \int_a^{\sigma(b)} G(\sigma(s), s) p(s) \Delta s \|u\|_\infty \right]^{-1}, \left[ \int_a^{\sigma(b)} G(\sigma(s), s) q(s) \Delta s \|u\|_\infty \right]^{-1} \right\}
\]
Theorem 3.1. Assume that conditions (A1) – (A4) are satisfied. Then, for each \( \lambda \) satisfying
\[
L_1 < \lambda < L_2,
\]
there exists a pair \((u, v)\) satisfying (1.1), (1.2) such that \( u(x) > 0 \) and \( v(x) > 0 \) on \((a, \sigma^3(b))\).

Proof. Let \( \lambda \) be as in (3.2). And let \( \epsilon > 0 \) be chosen such that
\[
\max \left\{ \frac{\gamma}{\lambda} \int_{a}^{b} G(t, s)p(s)\Delta s(f_\infty - \epsilon)^{-1}, \frac{\gamma}{\lambda} \int_{a}^{b} G(t, s)q(s)\Delta s(g_\infty - \epsilon)^{-1}\right\} \leq \lambda,
\]
\[
\lambda \leq \min \left\{ \int_{a}^{b} G(\sigma(s), s)p(s)\Delta s(f_\epsilon), \int_{b}^{\sigma(b)} G(\sigma(s), s)q(s)\Delta s(g_\epsilon)\right\}^{-1}.
\]
Define an integral operator \( T : P \rightarrow B \) by
\[
Tu(t) := \lambda \int_{a}^{b} G(t, s)p(s)f(\lambda \int_{a}^{b} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r)\Delta s, \quad u \in P.
\]

We seek suitable fixed points of \( T \) in the cone \( P \).

Notice from (A1), (A2), and (2.4) that, for \( u \in P \), \( Tu(t) \geq 0 \) on \([a, \sigma^3(b)]\). Also, for \( u \in P \), we have from (2.2) that
\[
Tu(t) := \lambda \int_{a}^{b} G(t, s)p(s)f(\lambda \int_{a}^{b} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r)\Delta s
\]
\[
\leq \lambda \int_{a}^{b} G(\sigma(s), s)p(s)f(\lambda \int_{a}^{b} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r)\Delta s
\]
so that
\[
\|Tu\| \leq \lambda \int_{a}^{b} G(\sigma(s), s)p(s)f(\lambda \int_{a}^{b} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r)\Delta s.
\]

Next, if \( u \in P \), we have from (2.2), (2.3), and (3.3) that
\[
\min_{t \in [a, b]} Tu(t) = \min_{t \in [a, b]} \lambda \int_{a}^{b} G(t, s)p(s)f(\lambda \int_{a}^{b} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r)\Delta s
\]
\[
\geq \lambda \gamma \int_{a}^{b} G(\sigma(s), s)p(s)f(\lambda \int_{a}^{b} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r)\Delta s
\]
\[
\geq \gamma \|Tu\|.
\]

Consequently, \( T : P \rightarrow P \). Moreover, \( T \) is completely continuous by a typical application of the Ascoli-Arzelà Theorem.

Now, from the definitions of \( f_0 \) and \( g_0 \), there exists \( H_1 > 0 \) such that
\[
f(x) \leq (f_0 + \epsilon)x, \quad g(x) \leq (g_0 + \epsilon)x, \quad 0 < x \leq H_1.
\]
Let $u \in \mathcal{P}$ with $\|u\| = H_1$. We first have from (2.2) and choice of $\epsilon$, for $a \leq s \leq \sigma(b)$, that
\[
\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r \leq \lambda \int_a^{\sigma(b)} G(\sigma(r), r)q(r)g(u(\sigma(r)))\Delta r \\
\leq \lambda \int_a^{\sigma(b)} G(\sigma(r), r)q(r)(g_0 + \epsilon)u(r)\Delta r \\
\leq \lambda \int_a^{\sigma(b)} G(\sigma(r), r)q(r)\Delta r (g_0 + \epsilon\|u\|) \\
\leq \|u\| = H_1.
\]
As a consequence, we next have from (2.2) and choice of $\epsilon$, for $a \leq t \leq \sigma^3(b)$, that
\[
Tu(t) = \lambda \int_a^{\sigma(b)} G(t, s)p(s)f(\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r) \Delta s \\
\leq \lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)(f_0 + \epsilon)\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r \Delta s \\
\leq \lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)(f_0 + \epsilon)H_1 \Delta s \\
\leq H_1 = \|u\|.
\]
So, $\|Tu\| \leq \|u\|$. If we set
\[
\Omega_1 = \{x \in B : \|x\| < H_1\},
\]
then
\[
\|Tu\| \leq \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial \Omega_1.
\]
(3.4)

Next, from the definitions of $f_{\infty}$ and $g_{\infty}$, there exists $\overline{H}_2 > 0$ such that
\[
f(x) \geq (f_{\infty} - \epsilon)x, \quad g(x) \geq (g_{\infty} - \epsilon)x, \quad x \geq \overline{H}_2.
\]
Let
\[
H_2 = \max\{2H_1, \frac{\overline{H}_2}{\gamma}\}.
\]
Let $u \in \mathcal{P}$ and $\|u\| = H_2$. Then,
\[
\min_{t \in [0, \omega]} u(t) \geq \gamma \|u\| \geq \overline{H}_2.
\]
Consequently, from (2.2) and choice of $\epsilon$, for $a \leq s \leq \sigma(b)$, we have that
\[
\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r \geq \lambda \int_\xi^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r \\
\geq \lambda \int_\xi^{\sigma(b)} G(\sigma(r), r)q(r)g(u(\sigma(r)))\Delta r \\
\geq \lambda \int_\xi^{\sigma(b)} G(\sigma(r), r)q(r)(g_{\infty} - \epsilon)u(r)\Delta r \\
\geq \gamma \lambda \int_\xi^{\sigma(b)} G(\sigma(r), r)(g_{\infty} - \epsilon)\Delta r \|u\| \\
\geq \|u\| = H_2.
\]
And so, we have from (2.2) and choice of $\epsilon$ that
\[
Tu(\tau) = \lambda \int_{a}^{\sigma(b)} G(\tau, s)p(s)f(\lambda \int_{a}^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r)\Delta s
\]
\[
\geq \lambda \int_{a}^{\sigma(b)} G(\tau, s)p(s)(f_\infty - \epsilon)\Delta \lambda \int_{a}^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r\Delta s
\]
\[
\geq \lambda \int_{a}^{\sigma(b)} G(\tau, s)p(s)(f_\infty - \epsilon)H_2\Delta s
\]
\[
\geq \gamma H_2 > H_2 = \|u\|.
\]

Hence, $\|Tu\| \geq \|u\|$. So if we set
\[
\Omega_2 = \{x \in \mathcal{B} : \|x\| < H_2\},
\]
then
\[
\|Tu\| \geq \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial \Omega_2.
\] (3.5)

Applying Theorem 2.2 to (3.4) and (3.5), we obtain that $T$ has a fixed point $u \in \mathcal{P} \cap (\overline{\Omega_2} \setminus \Omega_2)$. As such, and with $v$ being defined by
\[
v(t) = \lambda \int_{a}^{\sigma(b)} G(t, s)q(s)g(u(\sigma(s)))\Delta s,
\]
the pair $(u, v)$ is a desired solution of (1.1)-(1.2) for the given $\lambda$.

Prior to our next result, we introduce another hypothesis.

(A5) $g(0) = 0$, and $f$ is an increasing function.

We now define positive numbers $L_3$ and $L_4$ by
\[
L_3 : = \max\left\{\gamma \int_{\xi}^{\eta} G(\tau, s)p(s)\Delta s f_0, \gamma \int_{\xi}^{\eta} G(\tau, s)q(s)\Delta s g_0\right\}^{-1},
\] \[
L_4 : = \min\left\{\int_{a}^{\sigma(b)} G(\sigma(s), s)p(s)\Delta s f_\infty, \int_{a}^{\sigma(b)} G(\sigma(s), s)q(s)\Delta s g_\infty\right\}^{-1}.
\]

Theorem 3.2. Assume that conditions (A1)-(A5) are satisfied. Then, for each $\lambda$ satisfying
\[
L_3 < \lambda < L_4,
\] (3.6)
there exists a pair $(u, v)$ satisfying (1.1)-(1.2) such that $u(x) > 0$ and $v(x) > 0$ on $(a, \sigma^3(b))_\mathbb{T}$.

Proof. Let $\lambda$ be as in (3.6). And let $\epsilon > 0$ be chosen such that
\[
\max\left\{\gamma \int_{\xi}^{\eta} G(\tau, s)p(s)\Delta s(\epsilon_0 - \epsilon), \gamma \int_{\xi}^{\eta} G(\tau, s)q(s)\Delta s(g_0 - \epsilon)\right\} \leq \lambda,
\] \[
\lambda \leq \min\left\{\int_{a}^{\sigma(b)} G(\sigma(s), s)p(s)\Delta s(f_\infty + \epsilon), \int_{a}^{\sigma(b)} G(\sigma(s), s)q(s)\Delta s(g_\infty + \epsilon)\right\}^{-1}.
\]

Let $T$ be the cone preserving, completely continuous operator that was defined by (3.3).
From the definitions of \( f_0 \) and \( g_0 \), there exists \( H_1 > 0 \) such that
\[
f(x) \geq (f_0 - \varepsilon)x, \quad g(x) \geq (g_0 - \varepsilon)x, \quad 0 < x \leq H_1
\]
Now, \( g(0) = 0 \), and so there exists \( 0 < H_2 < H_1 \) such that
\[
\lambda g(x) \leq \frac{H_1}{\int_a^{\sigma(b)} G(\sigma(s), s) q(s) \Delta s}, \quad 0 \leq x \leq H_2.
\]
Choose \( u \in \mathcal{P} \) with \( \|u\| = H_2 \). Then, for \( a \leq s \leq \sigma(b) \), we have
\[
\lambda \int_a^{\sigma(b)} G(\sigma(s), r) q(r) g(u(\sigma(r))) \Delta r \leq \frac{\int_a^{\sigma(b)} G(\sigma(s), r) q(r) H_1 \Delta r}{\int_a^{\sigma(b)} G(\sigma(s), s) q(s) \Delta s} \leq H_1.
\]
Then,
\[
Tu(\tau) = \lambda \int_a^{\sigma(b)} G(\tau, s) p(s) f(\lambda \int_a^{\sigma(b)} G(\sigma(s), r) q(r) g(u(\sigma(r))) \Delta r) \Delta s
\]
\[
\geq \lambda \int_a^\eta G(\tau, s) p(s) (f_0 - \varepsilon) \lambda \int_a^{\sigma(b)} G(\sigma(s), r) q(r) g(u(\sigma(r))) \Delta r \Delta s
\]
\[
\geq \lambda \int_a^\eta G(\tau, s) p(s) (f_0 - \varepsilon) \lambda \int_a^\eta G(\tau, r) q(r) g(u(\sigma(r))) \Delta r \Delta s
\]
\[
\geq \lambda \int_a^\eta G(\tau, s) p(s) (f_0 - \varepsilon) \lambda \gamma \int_a^\eta G(\tau, r) q(r) (g_0 - \varepsilon) \|u\| \Delta r \Delta s
\]
\[
\geq \lambda \int_a^\eta G(\tau, s) p(s) (f_0 - \varepsilon) \|u\| \Delta s
\]
\[
\geq \lambda \gamma \int_a^\eta G(\tau, s) p(s) (f_0 - \varepsilon) \|u\| \Delta s \geq \|u\|.
\]
So, \( \|Tu\| \geq \|u\| \). If we put
\[
\Omega_1 = \{x \in \mathcal{B} : \|x\| < H_2\},
\]
then
\[
\|Tu\| \geq \|u\|, \quad \text{for} \quad u \in \mathcal{P} \cap \partial \Omega_1.
\] (3.7)

Next, by definitions of \( f_\infty \) and \( g_\infty \), there exists \( \Omega_1 \) such that
\[
f(x) \leq (f_0 - \varepsilon)x, \quad g(x) \leq (g_0 - \varepsilon)x, \quad x \geq \Omega_1
\]
There are two cases: (i) \( g \) is bounded, and (ii) \( g \) is unbounded.

For case (i), suppose \( N > 0 \) is such that \( g(x) \leq N \) for all \( 0 < x < \infty \). Then, for \( a \leq s \leq \sigma(b) \) and \( u \in \mathcal{P} \),
\[
\lambda \int_a^{\sigma(b)} G(\sigma(s), r) q(r) g(u(\sigma(r))) \Delta r \leq N \lambda \int_a^{\sigma(b)} G(\sigma(r), r) q(r) \Delta r.
\]
Let
\[
M = \max \left\{ f(x) \mid 0 \leq x \leq N \lambda \int_a^{\sigma(b)} G(\sigma(r), r) q(r) \Delta r \right\},
\]
and let

\[ H_3 > \max \left\{ 2H_2, M \lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)\Delta s \right\}. \]

Then, for \( u \in \mathcal{P} \) with \( \| u \| = H_3 \),

\[ Tu(t) \leq \lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)M\Delta s \leq H_3 = \| u \| \]

so that \( \| Tu \| \leq \| u \| \). If \( \Omega_2 = \{ x \in \mathcal{B} : \| x \| < H_3 \} \),

then

\[ \| Tu \| \leq \| u \|, \quad \text{for } u \in \mathcal{P} \cap \partial \Omega_2. \quad (3.8) \]

For case (ii), there exists \( H_3 > \max 2H_2, \overline{\Pi}_1 \) such that \( g(x) \leq g(H_3) \), for \( 0 < x \leq H_3 \). Similarly, there exists \( H_4 > \max H_3, \lambda \int_a^{\sigma(b)} G(\sigma(r), r)q(r)g(H_3)\Delta r \) such that \( f(x) \leq f(H_4) \), for \( 0 < x \leq H_4 \). Choosing \( u \in \mathcal{P} \) with \( \| u \| = H_4 \) we have

\[ Tu(t) \leq \lambda \int_a^{\sigma(b)} G(t, s)p(s)f(\lambda \int_a^{\sigma(b)} G(\sigma(r), r)q(r)g(H_3)\Delta r)\Delta s \leq \lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)\Delta s(f_\infty + \epsilon)H_4 \leq H_4 = \| u \|, \]

and so \( \| Tu \| \leq \| u \| \). For this case, if we let

\[ \Omega_2 = \{ x \in \mathcal{B} : \| x \| < H_4 \}, \]

then

\[ \| Tu \| \leq \| u \|, \quad \text{for } u \in \mathcal{P} \cap \partial \Omega_2. \quad (3.9) \]

In either cases, application of part (ii) of Theorem 2.2 yields a fixed point \( u \) of \( T \) belonging to \( \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1) \), which in turn yields a pair \( (u, v) \) satisfying (1.1)-(1.2) for the chosen value of \( \lambda \). \( \square \)

4. Example

In this section, we give an example illustrating our result. For the sake of simplicity we take \( \alpha = \frac{1}{5}, \beta = \frac{1}{3}, \xi = \frac{1}{4}, \eta = \frac{2}{5} \). Let \( \mathbb{T} = \{ 0 \} \cup \{ \frac{1}{n+1} : n \in \mathbb{N} \} \cup [1, 2] \). Consider the system of four-point dynamic equations

\[ u^{\Delta\Delta}(t) + \frac{1}{5} \lambda \frac{k e^u}{c + e^u + e^{2u}} = 0, t \in [0, 1] \cap \mathbb{T} \]

\[ v^{\Delta\Delta}(t) + \frac{1}{5} \lambda \frac{k e^v}{c + e^u + e^{2u}} = 0, t \in [0, 1] \cap \mathbb{T} \]
Positive Solution

\[ u(0) = u\left(\frac{1}{3}\right) = 0 = \frac{1}{2}u\left(\frac{2}{3}\right) - \frac{1}{3}u(\sigma^3(1)). \]
\[ v(0) = v\left(\frac{1}{3}\right) = 0 = \frac{1}{2}v\left(\frac{2}{3}\right) - \frac{1}{3}v(\sigma^3(1)). \]

Here \( p(t) = q(t) = \frac{1}{5}t, k = 200, c = 1000, \)
\[ f(v) = \frac{kve^v}{c + e^v + e^{2v}}, g(u) = \frac{kue^u}{c + e^u + e^{2u}}. \]

By simple calculation, we find \( \gamma = 0.06, f_0 = g_0 = \frac{k}{c+1} = \frac{400}{1025}, f_\infty = g_\infty = k = 400, L_1 = 1.3425, L_2 = 2.1698. \) Theorem 3.1, it follows that for every \( \lambda \) such that \( 1.3425 < \lambda < 2.1698, \) the four-point system of dynamic equation has at least one positive solution.

5. Conclusion

In this paper, we determined the eigenvalue intervals for which there exist positive solutions of the system of third order four-point bounder value problem on time scales which unifies the result on continuous intervals and discrete intervals by using A Guo-Krasnosel’kii fixed point theorem. These results are rapidly arising in the field of modeling and determination of flagellate protozoan in a viscous fluid in further research. Further, the study of time scale has led to several important applications, e.g., in the study of insect population models, neural networks, heat transfer and epidemic models.

Acknowledgement

The authors express their gratitude to the guidance of Prof. K. Rajendra Prasad and the referee for her\'s comments and suggestions.

References


