Some results for the weighted Drazin inverse of a modified matrix∗

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ABSTRACT
In this paper, we give some results for the W-weighted Drazin inverse of a modified matrix $M = A - CW_{d,w}WB$ in terms of the W-weighted Drazin inverse of the matrix $A$ and the generalized Schur complement $Z = D - BW_{d,w}WC$, generalizing some recent results in the literature.

Keywords: Drazin inverse; Weighted Drazin inverse; Modified matrix; Schur complement

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1. Introduction

Let $C^{m \times n}$ denote the set of $m \times n$ complex matrices. The Drazin inverse of $A \in C^{n \times n}$ is the unique matrix $X$, denoted by $A_d$, satisfying the following equations

$$A^{k+1}X = A^k, \quad XAX = X, \quad AX =XA,$$

where $k = \text{ind}(A)$ is the index of $A$, the smallest nonnegative integer for which $\text{rank}(A^{k+1}) = \text{rank}(A^k)$ (see[1-3]). In particular, when $\text{ind}(A) = 1$, the Drazin inverse of $A$ is called the group inverse of $A$ and is denoted by $A_g$. If $A$ is nonsingular, it is clearly $\text{ind}(A) = 0$ and $A^D = A^{-1}$. Throughout this paper, we denote by $A^w = I - AA_d$ and define $A^0 = I$, where $I$ is the identity matrix with proper sizes. In addition, the symbols $r(A)$ and $||A||$ will stand for rank and spectral norm of $A \in C^{m \times n}$.

Let $A \in C^{m \times n}, W \in C^{n \times m}$ with $\text{ind}(AW) = k$ and $X \in C^{m \times n}$ be a matrix such that

$$(AW)^{k+1}XW = (AW)^k, \quad XWAWX = X, \quad AWX = XWA,$$

then $X$ is called W-weighed Drazin inverse of $A$ and denoted by $X = A_{d,w}$ [4]. In particular, when $A$ is square matrix and $W = I$ then $A_{d,w} = A_d$.

The importance of the Drazin inverse (W-weighted Drazin inverse) and its applications are very useful which can be found in [1-13]. In 2006, Hartwig et al. [5] gave some expressions for the Drazin inverse and the W-weighted Drazin inverse in order to find the solution of a second-order differential equation

$$Ex''(t) + Fx'(t) + Gx(t) = 0.$$


This paper is organized as follows. In section 2, we give some results for the W-weighted Drazin inverse of the modified matrix $M = A - CWD_{d,w}WB$ in terms of the W-weighted drazin inverse of the matrix $A$ and the generalized Schur complement $Z = D - BW A_{d,w}WC$. Some relative results in [10,11,17] are the corollaries of our paper.

2. The W-weighted Drazin inverse of a modified matrix

In this section, we present some results for the W-weighted Drazin inverse of the modified matrix $M = A - CWD_{d,w}WB$ in terms of the W-weighted drazin inverse of the matrix $A$ and the generalized Schur complement $Z = D - BW A_{d,w}WC$. As a result, some conclusions in [10,11,17] are obtained directly from our results.

Let $A, B, C, D \in \mathbb{C}^{m \times n}$, $W \in \mathbb{C}^{n \times m}$. Throughout this paper, we adopt the following notations:

\[
K = A_{d,w}WC, \quad H = BW A_{d,w}, \quad \Gamma = HWK, \quad (2.1)
\]

\[
P = (I - AW A_{d,w}W)C, \quad Q = B(I - W A_{d,w}WA). \quad (2.2)
\]

**Theorem 2.1.** Suppose $P = 0$, $Q = 0$, $C(I - WD_{d,w}WD)WZ_{d,w}WB = 0$, $CW D_{d,w}W(I - ZWZ_{d,w}W)B = 0$, $C(I - WZ_{d,w}WZ)WD_{d,w}WB = 0$ and $CW Z_{d,w}W(I - DW D_{d,w}WB)B = 0$, then

\[
M_{d,w} = A_{d,w} + K W Z_{d,w}W H. \quad (2.3)
\]

**Proof.** Let the right hand side of (2.3) be $X$. Since

\[
MWX = (AW - CW D_{d,w}WBW)(A_{d,w} + KW Z_{d,w}WH)
\]

\[
= AW A_{d,w} + AW KW Z_{d,w}WH - CW D_{d,w}WBW A_{d,w}
\]

\[
- CW D_{d,w}WBW KW Z_{d,w}WH
\]

\[
= AW A_{d,w} + CW Z_{d,w}WH - CW D_{d,w}WH
\]

\[
- CW D_{d,w}W(D - Z)W Z_{d,w}WH
\]

\[
= AW A_{d,w} + C(I - WD_{d,w}WD)WZ_{d,w}W WB A_{d,w}
\]

\[
- CW D_{d,w}W(I - ZW Z_{d,w}W)BW A_{d,w}
\]

\[
= AW A_{d,w}
\]

and

\[
XMW = (A_{d,w} + KW Z_{d,w}WH)(WA - WCW D_{d,w}WB)
\]

\[
= A_{d,w} WA - A_{d,w} WCW D_{d,w}WB + KW Z_{d,w}W HW A
\]

\[
- KW Z_{d,w}W HW CW D_{d,w}WB
\]

\[
= A_{d,w} WA - KW D_{d,w}WB + KW Z_{d,w}WB
\]

\[
- KW Z_{d,w}W(D - Z)WD_{d,w}WB
\]

\[
= A_{d,w} WA - A_{d,w} WC(I - WZ_{d,w}WZ)WD_{d,w}WB
\]

\[
+ A_{d,w} WCW Z_{d,w}W(I - DW D_{d,w}WB)B
\]

\[
= A_{d,w} WA.
\]
Thus
\[ MWX = XWM. \tag{2.4} \]

While
\[
XWMWX = A_{d,w}WAW(A_{d,w} + KWZ_{d,w}WH)
= A_{d,w}WAWA_{d,w} + A_{d,w}WAWKWZ_{d,w}WH
= A_{d,w} + KWZ_{d,w}WH
= X.
\]

Finally, by induction we will prove that
\[
(MW)^{k+1}XW = (MW)^k,
\tag{2.5}
\]
where \( k \geq l = \text{Ind}(AW) \). For the case \( l = \text{Ind}(AW) = 1 \), it is easy to see from \((AW)^2A_{d,w}W = AW\) that
\[
(MW)^2XW = MWXMW
= (A - CW_{d,w}W)AWA_{d,w}W
= AW - CW_{d,w}WBW
= MW.
\]

Generally, for \( l = \text{Ind}(AW) > 1 \), note the fact \((AW)^lA_{d,w}W = (AW)^l\) that
\[
(MW)^{l+1}XW
= (MW)^lMWXW
= [(A - CW_{d,w}W)W]^lA_{d,w}W
= A - CW_{d,w}WBW(A - CW_{d,w}W)W \cdots
\times (A - CW_{d,w}W)WAWA_{d,w}W
= (I - CW_{d,w}WBW)A_{d,w}W(I - CW_{d,w}WBW)A_{d,w}W \cdots
\times (I - CW_{d,w}WBW)A_{d,w}WAWA_{d,w}W
= (I - CW_{d,w}WBW)A_{d,w}W[I - AWCW_{d,w}WBW(A_{d,w}W)^2](AW)^2 \cdots
\times (I - CW_{d,w}WBW)A_{d,w}WAWA_{d,w}W
= \cdots
= (I - CW_{d,w}WBW)A_{d,w}W[I - AWCW_{d,w}WBW(A_{d,w}W)^2] \cdots
\times (I - AW)(AW)^{l-1}WAWA_{d,w}W
= (I - CW_{d,w}WBW)A_{d,w}W[I - AWCW_{d,w}WBW(A_{d,w}W)^2] \cdots
\times (I - AW)(AW)^{l-1}WAWA_{d,w}W
= (I - CW_{d,w}WBW)A_{d,w}W[I - AWCW_{d,w}WBW(A_{d,w}W)^2] \cdots
\times (I - AW)(AW)^{l-1}WAWA_{d,w}W
= \cdots
= (AW - CW_{d,w}WBW)A_{d,w}WAW
\times (AW - CW_{d,w}WBW)A_{d,w}WAW
= [(A - CW_{d,w}W)W]^l
= (MW)^l.
\]
Corollary 2.1 ([17]). Let $A$, $B$, $C$, $D \in \mathbb{C}^{n \times n}$, and $W = I$ in (2.1) and (2.2). Suppose $P = 0$, $Q = 0$, $C(I - DD_d)Z_dB = 0$, $CD_d(I - ZZ_d)B = 0$, $C(I - ZZ_d)D_dB = 0$ and $CZ_d(I - DD_d)B = 0$, then

$$M_d = A_d + KZ_dH.$$ 

Specially, when $D = I$, we get

$$M_d = A_d + KZ_dH.$$ 

Moreover, if $Z$ is nonsingular, then

$$M_d = A_d + KZ^{-1}H.$$ 

From Corollary 2.1, when $C = I$, we get a result of perturbation of the Drazin inverse.

Corollary 2.2 ([10]). Suppose $B(I - AA_d) = 0$, $(I - AA_d)D_d = 0$ and $\|A_d\| \cdot \|D_dB\| \leq 1$, then

$$(A - D_dB)_d = (I - A_dD_dB)^{-1}A_d = A_d(I - D_dBA_d)^{-1}$$

and

$$(A - D_dB)_d - A_d = (A - D_dB)_dD_dBA_d = A_dD_dB(A - D_dB)_d,$$

with

$$\frac{\| (A - D_dB)_d - A_d \|}{\| A_d \|} \leq \frac{k_d(A) \| D_dB \| / \| A \|}{1 - k_d(A) \| D_dB \| / \| A \|},$$

where $k_d(A) = \| A \| / \| A_d \|$ is the condition number with respect to the Drazin inverse.

Theorem 2.2. Suppose $P = 0$, $Q = 0$, $Z = 0$, $C(I - WD_dW)WT_dW = 0$, $CD_dW(I - \Gamma WT_dW)B = 0$, $C(I - WT_dW)WT_dW = 0$ and $CWT_dW(I - DWd_dW)B = 0$, then

$$M_{d,w} = (I - KWT_{d,w}W)A_{d,w}(I - WKWT_{d,w}WH).$$

(2.6)

Proof. Let the right hand side of (2.6) be $X$. Firstly, we have

$$MWX = (A - CW \Gamma WT_{d,w}W)W(I - KWT_{d,w}W)A_{d,w}(I - WKWT_{d,w}WH)$$

$$= (AW - CW \Gamma WT_{d,w}W)A_{d,w} - A_{d,w}WKW \Gamma WT_{d,w}W$$

$$- KWT_{d,w}WH A_{d,w} + KWT_{d,w}WH A_{d,w}WKW \Gamma WT_{d,w}WH$$

$$= AW A_{d,w} - AW A_{d,w}WKW \Gamma WT_{d,w}WH - AWKW \Gamma WT_{d,w}WHA_{d,w}$$

$$+ AWKW \Gamma WT_{d,w}WHW A_{d,w}WKW \Gamma WT_{d,w}WH - CWD_{d,w}WBW A_{d,w}$$

$$+ CWD_{d,w}WBW A_{d,w}WKW \Gamma WT_{d,w}WH + CWD_{d,w}WBW KW \Gamma WT_{d,w}WH$$

$$\times WHW A_{d,w} - CWD_{d,w}WBWKW \Gamma WT_{d,w}WH A_{d,w}WKW \Gamma WT_{d,w}WH$$
\begin{align*}
&= AWA_{d,w} - KW_{d,w}WH - CW_{d,w}WHWA_{d,w} \\
&\quad + CW_{d,w}WHW_{d,w}WH - CW_{d,w}BW_{A_{d,w}} \\
&\quad + CW_{d,w}WHW_{d,w}WH - CW_{d,w}WD_{d,w}WHW_{A_{d,w}} \\
&\quad - CW_{d,w}WD_{d,w}WHWA_{d,w}WHWA_{d,w}W \\
&= AW_{A_{d,w}} - KW_{d,w}WH
\end{align*}

and

\begin{align*}
XWM &= (I - KW_{d,w}WHW)A_{d,w}(I - KW_{d,w}WH)W(A - CW_{d,w}WB) \\
&= (A_{d,w} - A_{d,w}KW_{d,w}WH - KW_{d,w}WHWA_{d,w}) \\
&\quad + KW_{d,w}WHWA_{d,w}KW_{d,w}WHW_{A_{d,w}W} \\
&\quad + KW_{d,w}WHW_{d,w}WHWA_{d,w}KW_{d,w}WHW_{A_{d,w}W} \\
&\quad \times WHASA_{d,w}KW_{d,w}WHWA_{d,w}KW_{d,w}WHW_{d,w}WB \\
&= A_{d,w}WA_{d,w} - A_{d,w}KW_{d,w}WHWA_{d,w}W \\
&\quad + A_{d,w}KW_{d,w}WHW_{d,w}WB - KW_{d,w}WHWA_{d,w}WB \\
&\quad + KW_{d,w}WHW_{d,w}WB - KW_{d,w}WHWA_{d,w}WB \\
&\quad - KW_{d,w}WHWA_{d,w}KW_{d,w}WHW_{d,w}WB \\
&= A_{d,w}WA_{d,w} - KW_{d,w}WH,
\end{align*}

i.e.,

\begin{equation}
MWX = XWM.
\end{equation}

Secondly, we get

\begin{align*}
XWM &= (A_{d,w}WA_{d,w} - KW_{d,w}WH)W(I - KW_{d,w}WH)A_{d,w} \\
&\times (I - KW_{d,w}WH) \\
&= (A_{d,w}WA_{d,w}AW - A_{d,w}AWKW_{d,w}WH - KW_{d,w}WHW_{A_{d,w}W} \\
&\quad + KW_{d,w}WHW_{d,w}WHWA_{d,w}KW_{d,w}WH)A_{d,w}(I - KW_{d,w}WH) \\
&= (I - KW_{d,w}WHW_{d,w}WBW_{A_{d,w}W}A_{d,w}W)A_{d,w}(I - KW_{d,w}WH) \\
&= (I - KW_{d,w}WHW_{d,w}WH)A_{d,w}(I - KW_{d,w}WH) \\
&= X.
\end{align*}

Finally, we shall prove that

\begin{equation}
(MW)^{k+1}XW = (MW)^k,
\end{equation}

by induction on $k \geq l = Ind(AW)$. For the case $l = Ind(AW) = 1$, it is easy to see from $(AW)^2A_{d,w}W = AW$ that

\begin{equation}
(MW)^2XW = MWXW
\end{equation}
For Corollary 2.4 (11). Suppose

\[ C \]

Theorem 2.3. Suppose

\[ (AW) \]

Therefore, (2.8) holds, which completes the proof. 

For \(l = Ind(AW) > 1\), note the fact \((AW)^{l+1}A_{d,w}W = (AW)^l\) that

\[
(MW)^{l+1}XW = (MW)^lWMXW
\]

\[
= \left[(A - CW D_{d,w} W B) W (AW A_{d,w} - K W T_{d,u} W H) W\right]
\]

\[
= \left[(A - CW D_{d,w} W B) W (AW A_{d,w} - K W T_{d,u} W H) W\right]
\]

\[
= \left[(I - A_{d,w} W C W T_{d,u} W H W)\right]
\]

\[
= \left[(A - CW D_{d,w} W B) W\right] - \left[(A - CW D_{d,w} W B) W\right] A_{d,w} W C W T_{d,u} W H W
\]

\[
= \left[(A - CW D_{d,w} W B) W\right] - \left[(A - CW D_{d,w} W B) W\right] A_{d,w} W C W T_{d,u} W H W
\]

\[
\times (A W A_{d,w} W C W T_{d,u} W H W - C W D_{d,w} W B A_{d,w} W C W T_{d,u} W H W)
\]

\[
= \left[(A - CW D_{d,w} W B) W\right] - \left[(A - CW D_{d,w} W B) W\right] A_{d,w} W C W T_{d,u} W H W
\]

\[
\times (C W T_{d,u} W H W - C W D_{d,w} W D W T_{d,u} W H W)
\]

\[
= \left[(A - CW D_{d,w} W B) W\right] - \left[(A - CW D_{d,w} W B) W\right] A_{d,w} W C W T_{d,u} W H W
\]

\[
= (MW)^l.
\]

For \(k \geq l = Ind(AW)\), it is easy to verify that

\[
(MW)^{k+1}XW = (MW)^k.
\]

Therefore, (2.8) holds, which completes the proof. 

By Theorem 2.2, when \(A, B, C, D\) are square and \(W = I\), we can directly get Theorem 2.2 in [17].

Corollary 2.3 ([17]). Suppose \(P = 0, Q = 0, Z = 0, C(I - D D_{d}) \Gamma_{d} B = 0, C D_{d} (I - \Gamma_{d}) B = 0, C(I - \Gamma_{d}) D_{d} B = 0, B = 0, C T_{d} (I - D D_{d}) B = 0, B = 0\), then

\[
M_{d} = (I - K \Gamma_{d} H) A_{d}(I - K \Gamma_{d} H).
\]

By Corollary 2.3, when \(D = I\), we get Theorem 2.2 in [11].

Corollary 2.4 ([11]). Suppose \(P = 0, Q = 0, Z = 0, C(I - \Gamma_{d}) B = 0, B = 0\), then

\[
M_{d} = (A - C B)_{d} = (I - K \Gamma_{d} H) A_{d}(I - K \Gamma_{d} H).
\]

Next, we present another result of this paper.

Theorem 2.3. Suppose \(P = 0, Q = 0, Ind(ZW) = 1, C(I - W D_{d,w} W D) = 0, (I - D W D_{d,w} W) B = 0, C W D_{d,w} W (I - \Gamma W T_{d,u} W) = 0, (I - W T_{d,u} W T) W D_{d,w} W B = 0, W Z_{d,u} W Z W T_{d,u} W = W T_{d,u} W Z W Z_{d,u} W\), then

\[
M_{d,w} = [I - K W (I - Z_{d,w} W Z W) \Gamma_{d,w} W H W] A_{d,w} \times
\]
\[ [I - WKWT_{d,w}W(I - ZWZ_{d,w}W)H] + KWZ_{d,w}WH. \] (2.9)

Proof. Let the right hand side of (2.9) be \( X \). First, note the facts:

\[
MW[I - KW(I - Z_{d,w}WZW)\Gamma_{d,w}WHW] = MW - (AW - CWD_{d,w}WBW)(KWT_{d,w}WHW - KWZ_{d,w}WZW_{d,w}WHW) = MW - AWKWT_{d,w}WHW + AWKWZ_{d,w}WZW_{d,w}WHW + CWD_{d,w}WGW_KWZ_{d,w}WZW_{d,w}WHW = MW
\]

Similarly, we get

\[
[I - WKWT_{d,w}W(I - ZWZ_{d,w}W)H]WM = WM,
\]

Now, we have

\[ MWX \]

\[
= MW[I - KW(I - Z_{d,w}WZW)\Gamma_{d,w}WHW]A_{d,w} \times [I - WKWT_{d,w}W(I - ZWZ_{d,w}W)H]AWD_{d,w}WGWKWT_{d,w}W(I - ZWZ_{d,w}W)H + CWD_{d,w}WZW_{d,w}WH - CWD_{d,w}WZW_{d,w}WH = AWAD_{d,w} - KWT_{d,w}WH + KWT_{d,w}WZW_{d,w}WH - CWD_{d,w}WZW_{d,w}WH + CWD_{d,w}WTWT_{d,w}WH = AWAD_{d,w} - KWT_{d,w}WH + KWT_{d,w}WZW_{d,w}WH - CWD_{d,w}WZW_{d,w}WH \times (I - CWT_{d,w}W)H + CWD_{d,w}W(I - CWT_{d,w}W)ZWZ_{d,w}WH = AWAD_{d,w} - KWT_{d,w}WH + KWT_{d,w}WZW_{d,w}WH \]

And

\[ XWM \]

\[
= [I - KW(I - Z_{d,w}WZW)\Gamma_{d,w}WHW]A_{d,w} \times [I - WKWT_{d,w}W(I - ZWZ_{d,w}W)H]WM + KWZ_{d,w}WHM = [I - KW(I - Z_{d,w}WZW)\Gamma_{d,w}WHW]A_{d,w}(WA - CWD_{d,w}WB) + KWZ_{d,w}WH(WA - CWD_{d,w}WB) = A_{d,w}WA - KW(I - Z_{d,w}WZW)\Gamma_{d,w}WHWA_{d,w}WA - A_{d,w}CWD_{d,w}WB + KW(I - Z_{d,w}WZW)\Gamma_{d,w}WHWA_{d,w}W \times CWD_{d,w}WB + KWZ_{d,w}WHWA - KWZ_{d,w}WHCWD_{d,w}WB = A_{d,w}WA - KWT_{d,w}WH + KWZ_{d,w}WZW_{d,w}WH - CWD_{d,w}WZW_{d,w}WH + KWT_{d,w}WTWT_{d,w}WB - KWZ_{d,w}WZW_{d,w}WTWD_{d,w}WB
\]
\[ +KWZ_{d,w}WB - KWZ_{d,w}WDW_{d,w}WB + KWZ_{d,w}ZW_{d,w}WD_{d,w}WB \]
\[ = A_{d,w}WA - KWT_{d,w}WH + KWZ_{d,w}ZW_{d,w}WT_{d,w}WH - K(I - WT_{d,w}WT) \]
\[ \times WD_{d,w}WB + KWZ_{d,w}ZW(I - WT_{d,w}WT)WD_{d,w}WB \]
\[ = A_{d,w}WA - KWT_{d,w}WH + KWZ_{d,w}ZW_{d,w}WH, \]
i.e.,
\[ MWX = XWM. \]  

(2.10)

Secondly, we get
\[ XWMWX = (A_{d,w}WA - KW_{d,w}WH + KWZ_{d,w}ZW_{d,w}WH)W \]
\[ \times [I - KW(I - Z_{d,w}ZW_{d,w})W \Gamma_{d,w} WHW]_{d,w} \]
\[ \times [I - WKWT_{d,w}W(I - ZW_{d,w}W)H] + (A_{d,w}WA \]
\[ -KW_{d,w}WH + KWZ_{d,w}ZW_{d,w}WH_{d,w}W] \]
\[ = (I - KW_{d,w}WH + KWZ_{d,w}ZW_{d,w}WH) \]
\[ \times [I - KW(I - Z_{d,w}ZW_{d,w})W \Gamma_{d,w} WHW]_{d,w} \]
\[ \times [I - WKWT_{d,w}W(I - ZW_{d,w}W)H] + KWZ_{d,w}WH \]
\[ = (I - KW_{d,w}WH + KWZ_{d,w}ZW_{d,w}WH) \]
\[ \times [I - WK(I - Z_{d,w}ZW_{d,w})W \Gamma_{d,w} WHW]_{d,w} \]
\[ \times [I - WKWT_{d,w}W(I - ZW_{d,w}W)H] + KWZ_{d,w}WH \]
\[ = X. \]

Finally, we shall prove that
\[ (MW)^{k+1}XW = (MW)^k, \]  
(2.11)

by induction on \( k \geq l = Ind(AW) \). For \( l = Ind(AW) \), note the facts:
\[ MW(I - KW(I - Z_{d,w}ZW_{d,w})W \Gamma_{d,w} WHW) = MW \]

and
\[ (MW)^lAW_{d,w}W = (MW)^l. \]

Now, we have
\[ (MW)^{l+1}XW = (MW)^lMWXW \]
\[ = (MW)^l(AW_{d,w}W - KWT_{d,w}WH + KWZ_{d,w}ZW_{d,w}W) \]
\[ = (MW)^l(I - KWT_{d,w}WH + KWZ_{d,w}ZW_{d,w}W)WAW_{d,w}W \]
\[ = (MW)^{l-1}[MW(I - KWT_{d,w}WH + KWZ_{d,w}ZW_{d,w}W)]WAW_{d,w}W \]
\[ = (MW)^{l-1}MW_{d,w}AW \]
\[ = (MW)^l. \]  
\[ (2.12) \]

For \( k \geq l = Ind(AW) \). From (2.12), we get (2.11), which completes the proof. \( \square \)
When $A$, $B$, $C$, $D$ are square and $W = I$, we get the following corollary.

**Corollary 2.5.** Suppose $P = 0$, $Q = 0$, $\text{Ind}(Z) = 1$, $C(I - DD_d) = 0$, $(I - DD_d)B = 0$, $CD_d(I - \Gamma_d) = 0$, $(I - \Gamma_d\Gamma)B = 0$ and $Z_dZ\Gamma_d = \Gamma_dZZ_d$, then

$$M_d = [I - K(I - ZZ_d)\Gamma_dH]A_d[I - K\Gamma_d(I - ZZ_d)H] + KZ_dH.$$  

By Corollary 2.5, when $D = I$, we have the following result.

**Corollary 2.6.** Suppose $P = 0$, $Q = 0$, $\text{Ind}(Z) = 1$, $C(I - \Gamma_d\Gamma) = 0$, $(I - \Gamma_d\Gamma)B = 0$ and $Z_dZ\Gamma_d = \Gamma_dZZ_d$, then


**References**